Math 2213 Introduction to Analysis

Homework 1 Due September 10 (Thursday), 2025

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Below is the definition of a metric from the lecture notes.

Definition (metric). A function $d : X \times X \to [0, \infty)$ is called a metric on X if, for all $x, y, z \in X$, the following properties hold:

(i) For any $x \in X$, we have $d(x, x) = 0$.

(ii) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(iii) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(iv) (Triangle Inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Problem 1 ((10 pts) Dyadic density via the Archimedean property)**.**

Let $a < b$ be real numbers. Prove that there exists a dyadic rational

$$q = \frac{k}{2^n} \in \mathbb{Q} \qquad (k \in \mathbb{Z}, n \in \mathbb{N})$$

such that $a < q < b$. Further show that there are infinitely many such dyadic rationals in (a, b).

Solution 1. Let $L = b - a > 0$. Notice that $2^n > n$ for all natural numbers $n \geq 1$, which derives from induction as follows: $2^1 > 1$ and $2^{n+1} = 2 \cdot 2^n > 2 \cdot n = n + n > n + 1$ for all $n \geq 1$. By the Archimedean property, there exists a natural number $n \geq 1$ such that $2^n L > nL > 1$, hence $\frac{1}{2^n} < L$.

Let $S_n = \{m \in \mathbb{Z} \mid m > 2^n a\}$. Since S_n is a nonempty set of integers bounded from below, it has a minimal element, say $k = \inf(S_n) \in \mathbb{Z}$. Then we have $k > 2^n a$, $k - 1 \leq 2^n a$, $2^n a + 1 < 2^n b$, so

$$2^n a < k \leq 2^n a + 1 < 2^n b.$$

Dividing by 2^n gives the desired dyadic rational.

To show that there are infinitely many such dyadic rationals in (a, b), we note that we can take any natural number $n' \geq n$, where n is some natural number satisfying $1/2^n < L$ found above. Then by the same argument, we can find a dyadic rational $q' = k'/2^{n'} \in (a, b)$, where $k' = \inf(\{m \in \mathbb{Z} \mid m > 2^{n'} a\})$. Since there are infinitely many natural numbers, hence infinitely many choices of n', there are infinitely many such dyadic rationals in (a, b).

Problem 2 (A tour of the p-adic world)**.**

The field $\mathbb{Q}$ inherits the Euclidean metric from $\mathbb{R}$, but it also carries a very different metric: the p-adic metric. Given a prime number p and an integer n, the p-adic norm of n is defined as

$$|n|_p = \frac{1}{p^k},$$

where p^k is the largest power of p dividing n. (We define $|0|_p := 0$.) The more factors of p appear in n, the smaller the p-adic norm becomes.

For a rational number $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, we may factor x as

$$x = p^k \cdot \frac{r}{s},$$

where $k \in \mathbb{Z}$ and p divides neither r nor s. We then define

$$|x|_p = p^{-k}.$$

The p-adic metric on $\mathbb{Q}$ is given by

$$d_p(x, y) := |x - y|_p.$$

1. To compute the 5-adic norm $|x|_5$ of a rational number x, we examine how many factors of 5 occur in x (in either numerator or denominator). If $x = 5^k \cdot \frac{a}{b}$ with a, b not divisible by 5 and $k \in \mathbb{Z}$, then the 5-adic norm is

$$|x|_5 = 5^{-k}.$$

   For example:

   (a) $30 = 2 \cdot 3 \cdot 5$. There is exactly one factor of 5, so

$$|30|_5 = 5^{-1} = \tfrac{1}{5}.$$

   (b) $32 = 2^5$. There is no factor of 5, so

$$|32|_5 = 5^0 = 1.$$

   (c) Compute $\left|\frac{1}{250}\right|_5$.

$$250 = 2 \cdot 5^3.$$

   So

$$\frac{1}{250} = \frac{1}{2 \cdot 5^3} = 5^{-3} \cdot \frac{1}{2},$$

   where $\frac{1}{2}$ has no factor of 5 in numerator or denominator. Therefore,

$$\left|\tfrac{1}{250}\right|_5 = 5^{-(-3)} = 5^3 = 125.$$

   Hence,

$$\boxed{\left|\tfrac{1}{250}\right|_5 = 125.}$$

   Now practice computing the following 5-adic norms: (6 pts)

   (a) $|75|_5$

   (b) $\left|\frac{10}{9}\right|_5$

   (c) $\left|-\frac{20}{375}\right|_5$

2. (9 pts) Further properties of the 5-adic norm.

   (a) Determine all rational numbers x satisfying $|x|_5 \leq 1$.

   (b) Which rational numbers x satisfy $|x|_5 = 1$?

   (c) What is $\lim_{n\to\infty} 5^n$ in $(\mathbb{Q}, d_5)$ (the 5-adic metric)?
   Hint: Compute $d_5(5^n, 0)$.

3. (15 pts) **Non-Archimedean absolute value and metric.** Prove that $|\cdot|_p$ satisfies

$$|xy|_p = |x|_p |y|_p, \qquad |x + y|_p \leq \max\{|x|_p, |y|_p\},$$

   and show that d_p is a metric on $\mathbb{Q}$.

Solution 2.

1. (a) $|75|_5 = 5^{-2} = \frac{1}{25}$.
   (b) $\left|\frac{10}{9}\right|_5 = 5^{-1} = \frac{1}{5}$.
   (c) $\left|-\frac{20}{375}\right|_5 = 5^1 = 25$.
2. (a) Suppose $|x|_5 \leq 1 = 5^0$. Since

$$|x|_5 = 5^{-\#(\text{factors of 5 in reduced form})},$$

   there must be no factors of 5 in the denominator of x when written as a reduced fraction. Thus, $x = 5^l p/q$, where $l \geq 0$ and 5 does not divide either p or q.
   (b) This is the $l = 0$ case from above. So $x = p/q$, where 5 does not divide either p or q.
   (c) Notice that $5^n > n$ for all $n \geq 1$ by mathematical induction, since $5^1 \geq 1$ and $5^{n+1} = 5 \times 5^n \geq 5n \geq n + 1$. So for all $\epsilon > 0$, choose $N = 1/\epsilon$, then

$$d_5(5^n, 0) = |5^n|_5 = \frac{1}{5^n} < \frac{1}{n} < \epsilon$$

   whenever $n > N$. Thus, $\lim_{n\to\infty} 5^n = 0$ in $(\mathbb{Q}, d_5)$.
3. Suppose x and y can be expressed as $x = p^k \cdot \frac{m}{n}$ and $y = p^l \cdot \frac{u}{v}$, where $k, l \in \mathbb{Z}$, and $m, n, u, v \in \mathbb{Z}$ are not divisible by p. Then

$$|x|_p = p^{-k}, \quad |y|_p = p^{-l},$$

$$xy = p^{k+l} \cdot \frac{mu}{nv},$$

   and

$$|xy|_p = p^{-(k+l)} = p^{-k} \cdot p^{-l} = |x|_p \cdot |y|_p.$$

   Without loss of generality, assume $k \leq l$. Then

$$x + y = p^k \cdot \frac{m}{n} + p^l \cdot \frac{u}{v} = p^k \left(\frac{m}{n} + p^{l-k} \cdot \frac{u}{v}\right).$$

   Since $\frac{m}{n} + p^{l-k} \cdot \frac{u}{v}$ is not divisible by p, we have

$$|x + y|_p = p^{-k} = \max\{p^{-k}, p^{-l}\} = \max\{|x|_p, |y|_p\}.$$

   Finally, we verify that d_p is a metric on $\mathbb{Q}$ by checking the four properties of a metric:
   (i) For $x \in \mathbb{Q}$, we have $d_p(x, x) = |x - x|_p = |0| \equiv 0$.
   (ii) For $x, y \in \mathbb{Q}$ and $x \neq y$, we have $d_p(x, y) = |x - y|_p = a/b$ for some $a, b \in \mathbb{Z}, a \neq 0$.
   (iii) For $x, y \in \mathbb{Q}$, we have $d_p(x, y) = d_p(y, x)$.
   (iv) For $x, y, z \in \mathbb{Q}$, we have

$$\begin{aligned} d_p(x, z) &= |x - z|_p = |(x - y) + (y - z)|_p \\ &\leq \max\{|x - y|_p, |y - z|_p\} \\ &\leq |x - y|_p + |y - z|_p = d_p(x, y) + d_p(y, z), \end{aligned} \tag{1}$$

   since $\max\{a, b\} \leq a + b$ for all $a, b \geq 0$.

   Thus, d_p is a metric on $\mathbb{Q}$. Furthermore, it is a non-Archimedean metric since it satisfies the **strong triangle inequality** $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}$.

Problem 3 (Exercise 1.1.3 (20 pts)). Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.

(a) Give an example of a pair (X, d) which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)

(b) Give an example of a pair (X, d) which obeys axioms (acd) of Definition 1.1.2, but not (b).

(c) Give an example of a pair (X, d) which obeys axioms (abd) of Definition 1.1.2, but not (c).

(d) Give an example of a pair (X, d) which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where X is a finite set.)

Solution 3. Recall the definition of a metric. We shall give examples for each case below.

(a) Let $X = \mathbb{R}$ and define d such that $d(x, y) = 0.5$ if $x \neq y$ and $d(x, x) = 1$. By construction (a) is not satisfied. Furthermore, $d(x, y) = d(y, x) = 0.5$ for all distinct $x, y \in X$, so (b) and (c) are satisfied. Finally, for distinct $x, y, z \in X$, we have $d(x, z) = 0.5 \leq 0.5 + 0.5 = d(x, y) + d(y, z)$; for $x = z$, we have $d(x, z) = 1 \leq 0.5 + 0.5 = d(x, y) + d(y, z)$; for $x = y \neq z$, we have $d(x, z) = 0.5 \leq 1 + 0.5 = d(x, y) + d(y, z)$, so (d) is satisfied.

(b) Let $X = \mathbb{R}$ and $d(x, y) = 0$ for all $x, y \in X$. By construction (b) is not satisfied. Furthermore, $d(x, x) = 0$ for all $x \in X$, $d(x, y) = d(y, x) = 0$ for all $x, y \in X$, and $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$, for any $x, y, z \in X$, so (a), (c) and (d) are satisfied.

(c) Let $X = S^1$ the unit circle, and d the shortest clockwise distance between two points on the circle. Then $d(x, x) = 0$ for all $x \in X$ and $d(x, y) > 0$ for all distinct $x, y \in X$. Furthermore, for any $x, y, z \in X$, if y lies between x and z, then $d(x, z) = d(x, y) + d(y, z)$, and $d(x, z) < d(x, y) + d(y, z)$ otherwise. Thus (a), (b) and (d) are satisfied. However, unless x, y lie on the antipodal points of the circle, $d(x, y) \neq d(y, x)$, so (c) is not satisfied.

(d) Let $X = \mathbb{R}$ and $d(x, y) = (x - y)^2$. Then $d(x, x) = 0$ for all $x \in X$, $d(x, y) > 0$ for all distinct $x, y \in X$, and $d(x, y) = d(y, x)$ for all $x, y \in X$, so (a), (b) and (c) are satisfied. However, for $x = 0, y = 1, z = 2$, we have $d(x, z) = 4 \nleq 1 + 1 = d(x, y) + d(y, z)$, so (d) is not satisfied.

Problem 4 (20 pts). Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vectors in $\mathbb{R}^n$.

(a) The ℓ^1 metric is defined by

$$d_1(x, y) := \sum_{i=1}^{n} |x_i - y_i|.$$

Show that d_1 is a metric on $\mathbb{R}^n$

(b) The ℓ^∞ metric is defined by

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Show that d_∞ is a metric on $\mathbb{R}^n$

Solution 4.

(a) We verify the four properties of a metric:

(i) $d(x, x) = 0$

(ii) Each absolute value in the sum is non-negative. Moreover, if $x \neq y$, there must exist some i such that $x_i \neq y_i$, hence $d_1(x, y) > 0$.

(iii) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(y, x)$

(iv) By the triangle inequality of real numbers, we have

$$\begin{aligned} d_1(x,z) &= \sum_{i=1}^{n} |x_i - z_i| \\ &\leq \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) \\ &= \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| \\ &= d_1(x,y) + d_1(y,z). \end{aligned} \tag{2}$$

Hence d_1 is a metric on $\mathbb{R}^n$.

(b) We verify the four properties of a metric:

(i) $d_\infty(x,x) = 0$

(ii) Each absolute value in the maximum is non-negative. Moreover, if $x \neq y$, there must exist some i such that $x_i \neq y_i$, hence $d_\infty(x,y) > 0$.

(iii) $d_\infty(x,y) = \max_{1\leq i\leq n} |x_i - y_i| = \max_{1\leq i\leq n} |y_i - x_i| = d_\infty(y,x)$

(iv) By the triangle inequality of real numbers, we have

$$\begin{aligned} d_\infty(x,z) &= \max_{1\leq i\leq n} |x_i - z_i| \\ &\leq \max_{1\leq i\leq n} (|x_i - y_i| + |y_i - z_i|) \\ &\leq \max_{1\leq i\leq n} |x_i - y_i| + \max_{1\leq i\leq n} |y_i - z_i| \\ &= d_\infty(x,y) + d_\infty(y,z). \end{aligned} \tag{3}$$

Hence d_∞ is a metric on $\mathbb{R}^n$.

Problem 5 (10 pts). A vector space V over $\mathbb{R}$ s a set equipped with two operations:

1. **Vector addition:** $+ : V \times V \to V$, written $(u,v) \mapsto u + v$.
2. **Scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$, written $(\alpha, v) \mapsto \alpha v$,

such that the following properties hold for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

(VS1) $(u+v)+w = u+(v+w)$ (associativity of addition)

(VS2) $u+v = v+u$ (commutativity of addition)

(VS3) There exists $0 \in V$ such that $u + 0 = u$ (additive identity)

(VS4) For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0$ (additive inverse)

(VS5) $\alpha(u+v) = \alpha u + \alpha v$ (distributivity I)

(VS6) $(\alpha+\beta)u = \alpha u + \beta u$ (distributivity II)

(VS7) $(\alpha\beta)u = \alpha(\ betau)$ (compatibility of scalar multiplication)

(VS8) $1 \cdot u = u$ (identity element of scalar multiplication)

A function $\|\cdot\| : V \to [0, \infty)$ is called a norm on V if, for all $u, v \in V$ and $\alpha \in \mathbb{R}$, the following properties hold:

(N1) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$. (positivity)

(N2) $\|\alpha v\| = |\alpha| \cdot \|v\|$. (homogeneity)

(N3) $\|u+v\| \leq \|u\| + \|v\|$. (triangle inequality)

Given a norm $\|\cdot\|$ on V, define $d : V \times V \to [0, \infty)$ by

$$d(u, v) = \|u - v\|.$$

Prove that d is a metric on V, that is, for all $x, y, z \in V$ show that:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

(Thus we conclude that every normed vector space $(V, \|\cdot\|)$ is also a metric space with metric $d(u, v) = \|u - v\|$.)

Solution 5. We will show that the three properties of a metric are satisfied.

(i) The conditions that $d(x, y) = \|x - y\| \geq 0$ and $d(x, y) = \|x - y\| = 0$ if and only if $x = y$ are equivalent to (N1).

(ii) $d(x, y) = \|x - y\| = \|-(y - x)\| = \|(-1)(y - x)\| = |-1| \cdot \|y - x\| = d(y, x)$ by (N2).

(iii) $d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$ by (N3).

Thus every normed vector space $(V, \|\cdot\|)$ is also a metric space with metric $d(u, v) = \|u - v\|$.

Problem 6. Let S be a bounded nonempty set of real numbers, and let a and b be fixed nonzero real numbers. Define $T = \{as + b | s \in S\}$ Find formulas for $\sup T$ and $\inf T$ in terms of $\sup S$ and $\inf S$. Prove your formulas.

Solution 6.

Claim. The supremum and infimum of T are given by

$$\sup T = a \sup S + b, \quad \inf T = a \inf S + b. \tag{4}$$

Proof. Since S is a bounded nonempty set of real numbers, both $\sup S$ and $\inf S$ exist. We consider two cases based on the sign of a.

(a) If $a > 0$, then for all $s \in S$, we have $as + b \leq a \cdot \sup S + b$, so $a \cdot \sup S + b$ is an upper bound of T. By the definition of supremum, for any $\epsilon > 0$ there exists some $s' \in S$ such that $\sup S - \epsilon \leq s' \leq \sup S$. Multiplying by $a > 0$ and adding b gives

$$a \sup S + b - a\epsilon \leq as' + b \leq a \sup S + b.$$

Hence $\sup T = a \sup S + b$.

(b) If $a < 0$, similarly for all $s \in S$, we have $as + b \leq a \inf S + b$, so $a \inf S + b$ is an upper bound of T. By definition of the supremum, for any $\epsilon > 0$ there exists some $s' \in S$ such that $\inf S \leq s' < \inf S + \epsilon$. Multiplying by $a < 0$ and adding b gives

$$a \inf S + b \leq as' + b < a \inf S + b - a\epsilon.$$

Hence $\sup T = a \cdot \inf S + b$.

Since a, b are nonzero, we are done. □