Math 2213 Introduction to Analysis I

Homework 10 Due November 28 (Friday), 2025

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Corollary 1 (3.7.3)**.** Let $[a, b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Suppose that the series $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ is absolute convergent. Suppose also that $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent for some $x_0 \in [a, b]$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function $f : [a, b] \to \mathbb{R}$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x).$$

Exercise 1 (**Exercise 4.7.8**, 15 points)**.** Let $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ be the tangent function $\tan(x) := \sin(x)/\cos(x)$. Show that tan is differentiable and monotone increasing, with

$$\frac{d}{dx} \tan(x) = 1 + \tan(x)^2,$$

and that $\lim_{x\to\pi/2} \tan(x) = +\infty$ and $\lim_{x\to-\pi/2} \tan(x) = -\infty$. Conclude that tan is in fact a bijection from $(-\pi/2, \pi/2) \to \mathbb{R}$, and thus has an inverse function

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

(this function is called the <u>arctangent function</u>). Show that $\tan^{-1}$ is differentiable and

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}.$$

Solution 1. On $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\cos x > 0$, so $\tan x$ is defined on all of its domain and

$$\frac{d}{dx} \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x > 0.$$

Hence $\tan x$ is differentiable and monotone increasing. Now we show the limits of tan as $x \to \pm\frac{\pi}{2}$: Since sin is continuous and $\sin\frac{\pi}{2} = 1$, there exists $\delta_1 > 0$ such that $\sin x > \frac{1}{2}$ whenever $|x - \frac{\pi}{2}| < \delta_1$. Since cos is continuous and $\cos\frac{\pi}{2} = 0$, for any $\varepsilon > 0$ there exists $\delta_2 > 0$ such that $\cos x < \varepsilon$ whenever $|x - \frac{\pi}{2}| < \delta_2$. Let $M > 0$ be arbitray, $\varepsilon = \frac{1}{2M}$, and $\delta = \min\{\delta_1, \delta_2\}$. Then, for any x satisfying $0 < |x - \frac{\pi}{2}| < \delta$, we have

$$\tan x = \frac{\sin x}{\cos x} > \frac{\frac{1}{2}}{\varepsilon} = M \implies \lim_{x\to\frac{\pi}{2}} \tan x = +\infty.$$

By an analogous argument but with $\sin x < -\frac{1}{2}$ and $\cos x < \varepsilon$ for x close to $-\frac{\pi}{2}$, we have, for arbitrary $M > 0$, $\varepsilon = \frac{1}{2M}$, and $\tilde{\delta} = \min\{\tilde{\delta_1}, \tilde{\delta_2}\}$, that for any x satisfying $0 < |x + \frac{\pi}{2}| < \tilde{\delta}$,

$$\tan x = \frac{\sin x}{\cos x} < \frac{-\frac{1}{2}}{\varepsilon} = -M \implies \lim_{x\to-\frac{\pi}{2}} \tan x = -\infty.$$

Since tan is monotone increasing, it is injective. By the intermediate value theorem, it is also surjective onto $\mathbb{R}$. Thus tan is a bijection from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$, and has an inverse function $\tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Differentiating both sides of the identity $\tan\left(\tan^{-1} x\right) = x$, we have

$$\sec^2\left(\tan^{-1} x\right) \cdot \frac{d}{dx} \tan^{-1} x = 1,$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \cos^2\left(\tan^{-1}x\right) = \frac{1}{1+\tan^2\left(\tan^{-1}x\right)} = \frac{1}{1+x^2}.$$

Exercise 2 (**Exercise 4.7.9**, 15 points)**.** Recall the arctangent function $\tan^{-1}$ from Exercise 4.7.8. By modifying the proof of Theorem 4.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $x \in (-1, 1)$. Using Abel's theorem (Theorem 4.3.1) to extend this identity to the case $x = 1$, conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.11.) Conclude in particular that $4 - \frac{4}{3} < \pi < 4$. (One can of course compute $\pi = 3.1415926\ldots$ to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly.)

Solution 2. For $x \in (-1, 1)$, we have that for any $r < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2} \rightrightarrows \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

on $[-r, r]$. Since $\tan^{-1}(0) = 0$, integrating both sides from 0 to x, we have

$$\tan^{-1}x = \int_0^x \frac{\mathrm{d}t}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},$$

since $(-1)^n t^{2n}$ converges uniformly on compact subsets of $(-1, 1)$ and is Riemann integrable for each n. The resulting series converges by the alternating series test. Hence, by Abel's Theorem, we have

$$\frac{\pi}{4} = \tan^{-1}1 = \lim_{x\to 1^-}\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Therefore,

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

and $4 - \frac{4}{3} < \pi < 4$ since the series is alternating with decreasing terms.

Exercise 3 (**Exercise 4.7.10**, 30 points)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n}\cos(32^n \pi x).$$

(a) Show that this series is uniformly convergent, and that f is continuous.

(b) Show that for every integer j and every integer $m \geq 1$, we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)\right| \geq 4^{-m}.$$

Hint: use the identity

$$\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{m-1} a_n\right) + a_m + \sum_{n=m+1}^{\infty} a_n$$

for certain sequences a_n. Also, use the fact that the cosine function is periodic with period 2π, as well as the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for any $|r| < 1$. Finally, you will need the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ for any real numbers x and y; this can be proven by using the mean value theorem.

(c) Using (b), show that for every real number x_0, the function f is not differentiable at x_0. Hint: for every x_0 and every $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j+1$, thanks to Exercise 5.4.3.

(d) Explain briefly why the result in (c) does not contradict Corollary 3.7.3.

Solution 3.

(a) Since $|\cos(32^n \pi x)| \leq 1$, we have

$$|4^{-n} \cos(32^n \pi x)| \leq 4^{-n}.$$

The series $\sum_{n=1}^{\infty} 4^{-n}$ is a geometric series with ratio $\frac{1}{4}$, which converges. Hence, by the Weierstrass M-test, the series defining $f(x)$ converges uniformly. Since each term $4^{-n}\cos(32^n \pi x)$ is continuous, the uniform limit f is also continuous.

(b) We can write

$$f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) = \sum_{n=1}^{\infty} 4^{-n}\left[\cos\left(32^n \pi \frac{j+1}{32^m}\right) - \cos\left(32^n \pi \frac{j}{32^m}\right)\right].$$

For $n > m$, we have

$$\cos\left(32^n \pi \frac{j+1}{32^m}\right) = \cos\left(32^n \pi \frac{j}{32^m} + 32^{n-m}\pi\right) = \cos\left(32^n \pi \frac{j}{32^m}\right),$$

so we are left with only the first m terms, which can be split as

$$\begin{aligned} f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) &= \sum_{n=1}^{m-1} 4^{-n}\left[\cos\left(32^n \pi \frac{j+1}{32^m}\right) - \cos\left(32^n \pi \frac{j}{32^m}\right)\right] \\ &\quad + 4^{-m}\left[\cos\left(\pi(j+1)\right) - \cos\left(j\pi\right)\right] \\ &\equiv R_m(j) + 4^{-m}\left[\cos\left(\pi(j+1)\right) - \cos\left(j\pi\right)\right]. \end{aligned}$$

The first sum $R_m(j)$ can be bounded using the inequality $|\cos(x) - \cos(y)| \leq |x-y|$:

$$\begin{aligned} R_m(j) &= \left|\sum_{n=1}^{m-1} 4^{-n}\left[\cos\left(\pi \frac{j+1}{32^{m-n}}\right) - \cos\left(\pi \frac{j}{32^{m-n}}\right)\right]\right| \\ &\leq \sum_{n=1}^{m-1} 4^{-n}\left|\pi \frac{j+1}{32^{m-n}} - \pi \frac{j}{32^{m-n}}\right| \\ &= \sum_{n=1}^{m-1} \frac{4^{-n}\pi}{32^{m-n}} = \frac{\pi}{32^m}\sum_{n=1}^{m-1} 8^n \\ &= \frac{\pi}{32^m} \cdot \frac{8}{7}(8^{m-1} - 1) = \frac{\pi}{7}\left(4^{-m+1} - \frac{1}{32^m}\right) < \frac{4\pi}{7}4^{-m}. \end{aligned}$$

and since $|\cos((j+1)\pi) - \cos(j\pi)| = 2$, we have

$$\begin{aligned} \left|f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)\right| &\geq \left|4^{-m}\left[\cos\left(\pi(j+1)\right) - \cos\left(j\pi\right)\right]\right| - |R_m(j)| \\ &\geq 2 \cdot 4^{-m} - \frac{4\pi}{7}4^{-m} = \left(2 - \frac{4\pi}{7}\right)4^{-m} > 4^{-m}. \end{aligned}$$

(c) For $x_0 \in \mathbb{R}$, by Exercise 5.4.3, for each $m \geq 1$, there exists an integer j such that $j \leq 32^m x_0 \leq j+1$. Then,

$$\left| \frac{f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)}{\frac{j+1}{32^m} - \frac{j}{32^m}} \right| = 32^m \left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) \right|$$
$$\geq 32^m \cdot 4^{-m} = 8^m.$$

As $\frac{1}{32^m} \to 0$, or, $m \to \infty$, we have $8^m \to \infty$. Thus, the difference quotient does not converge, and by the definition of the derivative f is not differentiable at x_0.

(d) Refer to the statement of Corollary 3.7.3 at the beginning of this document. The Corollary requires that $\sum_{n=1}^{\infty} \|f_n'\|$ converges absolutely. However,

$$|f_n'| = |8^n \pi \sin(32^n \pi x)| \implies \|f_n'\|_\infty = \sup_{x\in\mathbb{R}} |8^n \pi \sin(32^n \pi x)| = 8^n \pi,$$

which is unbounded for $n \in \mathbb{N}$, and hence $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ does not converge absolutely. Therefore, the result in (c) does not contradict Corollary 3.7.3.

Exercise 4 (20 points)**.**

(a) Prove that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for all integers n and all real θ.This is the classical DeMoivre's theorem.

(b) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin n\theta = \sin^n\theta \left\{ \binom{n}{1} \cot^{n-1}\theta - \binom{n}{3} \cot^{n-3}\theta + \binom{n}{5} \cot^{n-5}\theta - \cdots \right\}.$$

(c) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta \; P_m(\cot^2\theta)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1} x^m - \binom{2m+1}{3} x^{m-1} + \binom{2m+1}{5} x^{m-2} - \cdots.$$

Use this to show that P_m has zeros at the m distinct points

$$x_k = \cot^2\left(\frac{\pi k}{2m+1}\right), \qquad k = 1, 2, \ldots, m.$$

(d) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

Solution 4.

(a) By Theorem 4.7.2 (f), for $\theta \in \mathbb{R}$ we have $e^{i\theta} = \cos\theta + i\sin\theta$. Raising both sides to the power n, we get $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} == \cos(n\theta) + i\sin(n\theta)$.

(b) Expanding $(\cos\theta + i\sin\theta)^n$ using the binomial theorem gives

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k = \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k}\theta \sin^k\theta.$$

The imaginary part is given by the sum over odd k, hence,

$$\begin{aligned}\sin(n\theta) &= \sum_{k=1,\, k \text{ odd}}^{n} \binom{n}{k} (-1)^{\frac{k-1}{2}} \cos^{n-k}\theta \sin^k\theta \\ &= \sin^n\theta \sum_{k=1,\, k \text{ odd}}^{n} \binom{n}{k} (-1)^{\frac{k-1}{2}} \cot^{n-k}\theta \\ &= \sin^n\theta \left\{ \binom{n}{1} \cot^{n-1}\theta - \binom{n}{3} \cot^{n-3}\theta + \binom{n}{5} \cot^{n-5}\theta - \cdots \right\}.\end{aligned}$$

(c) For $n = 2m+1$, we have

$$\sin(2m+1)\theta = \sin^{2m+1}\theta \left\{ \binom{2m+1}{1} \cot^{2m}\theta - \binom{2m+1}{3} \cot^{2m-2}\theta + \cdots \right\}$$

by the result of (b). Hence, by the definition of $P_m(x)$, we have

$$\sin(2m+1)\theta = \sin^{2m+1}\theta\, P_m(\cot^2\theta).$$

Since $0 < \theta < \frac{\pi}{2}$, we have $\sin(2m+1)\theta = 0$ when $\theta = \frac{\pi k}{2m+1}$ for $k = 1, 2, \ldots, m$. Note that at these points, $\sin\theta \neq 0$. Thus, $P_m(\cot^2\theta) = \sin(2m+1)\theta / \sin^{2m+1}\theta = 0$ at these points, so P_m has zeros at

$$x_k = \cot^2\left(\frac{\pi k}{2m+1}\right), \quad k = 1, 2, \ldots, m.$$

(d) By Vieta's formula (根與係數), the sum of the zeros of $P_m(x)$ is given by

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = -\left(-\binom{2m+1}{3}\right) \Big/ \binom{2m+1}{1} = \frac{m(2m-1)}{3}.$$

Exercise 5 (20 points). This exercise outlines a simple proof of the formula $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$. Start with the inequality

$$\sin x < x < \tan x, \qquad 0 < x < \frac{\pi}{2},$$

take reciprocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put $x = \dfrac{k\pi}{2m+1}$, where k and m are integers with $1 \le k \le m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right).$$

Use the formula in problem 4(d) to deduce the inequality

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}.$$

Now let $m \to \infty$ to obtain $\zeta(2) = \pi^2/6$.

Solution 5. Following the steps in the problem statement, we have

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{\pi^2}\sum_{k=1}^{m}\frac{1}{k^2} < m + \sum_{k=1}^{m}\cot^2\left(\frac{k\pi}{2m+1}\right).$$

By Exercise 4(d), we have

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2}\sum_{k=1}^{m}\frac{1}{k^2} < m + \frac{m(2m-1)}{3}.$$

Rearranging gives

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m}\frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}.$$

Take the limit as $m \to \infty$, we have

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} \to \frac{\pi^2}{6}, \quad \frac{2m(m+1)\pi^2}{3(2m+1)^2} \to \frac{\pi^2}{6},$$

hence by the Squeeze Theorem, we have that

$$\zeta(2) = \lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$