Math 2213 Introduction to Analysis I

Homework 11 Due December 5 (Friday), 2025

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Exercise 1 (**Exercise 5.2.6**, 20 points). Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$.

(a) Show that if f_n converges uniformly to f, then f_n also converges to f in the L^2 metric.

(b) Give an example where f_n converges to f in the L^2 metric, but does not converge to f uniformly. *(Hint: take $f = 0$. Try to make the functions f_n large in sup norm.)*

(c) Give an example where f_n converges to f in the L^2 metric, but does not converge to f pointwise. *(Hint: take $f = 0$. Try to make the functions f_n large at one point.)*

(d) Give an example where f_n converges to f pointwise, but does not converge to f in the L^2 metric. *(Hint: take $f = 0$. Try to make the functions f_n large in L^2 norm.)*

Solution 1.

(a) Suppose $f_n \rightrightarrows f$, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in \mathbb{R}/\mathbb{Z}$, we have $|f(x) - f_n(x)|_\infty < \varepsilon$ whenever $n > N$. Then, for $n > N$, we have

$$\|f_n - f\|_2 = \left(\int_0^1 \mathrm{d}t \, |f_n(t) - f(t)|^2\right)^{1/2} \leq \left(\int_0^1 \mathrm{d}t \, \varepsilon^2\right)^{1/2} = \varepsilon,$$

so $f_n \to f$ in the L^2 metric.

(b) Consider the sequence of functions $f_n(x) = \begin{cases} n, & x \in [0, \frac{1}{n^3}] \\ 0, & \text{otherwise} \end{cases}$. Then, for any $n \in \mathbb{N}$, we have

$$\|f_n - 0\|_2 = \left(\int_0^1 \mathrm{d}t \, |f_n(t) - 0|^2\right)^{1/2} = \left(\int_0^{1/n^3} \mathrm{d}t \, n^2\right)^{1/2} = 1/\sqrt{n} \to 0,$$

but f_n does not converge to 0 uniformly since $|f_n - f|_\infty = n \to \infty$ as $n \to \infty$.

(c) The same example as in (b) works here. We have $f_n \to f$ in the L^2 metric, but for $x = 0$, $|f_n(0) - 0| = n \to \infty$ as $n \to \infty$.

(d) Consider the sequence of functions $f_n(x) = \begin{cases} \sqrt{n}, & x \in [0, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$. Then $f(0) = 0$, and for any $x \in (0, 1]$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x \notin [0, \frac{1}{n}]$, so $f_n(x) = 0$. Thus, $f_n(x) \to 0$ pointwise. However, we have

$$\|f_n - 0\|_2 = \left(\int_0^1 \mathrm{d}t \, |f_n(t) - 0|^2\right)^{1/2} = \left(\int_0^{1/n} \mathrm{d}t \, n\right)^{1/2} = 1,$$

so f_n does not converge to 0 in the L^2 metric.

Exercise 2 (20 points). Let $\{\phi_N\} : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous, periodic functions on $\mathbb{R}$ (with period 1) which satisfy

$$\int_0^1 \phi_N(t) \, dt = 1 \qquad \text{and} \qquad \int_0^1 |\phi_N(t)| \, dt \leq M < \infty$$

for all $N \in \mathbb{N}$, and

$$\lim_{N\to\infty} \int_\delta^{1-\delta} |\phi_N(t)|\, dt = 0$$

for each $0 < \delta < 1$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1. Prove that

$$\lim_{N\to\infty} \int_0^1 f(x-t)\, \phi_N(t)\, dt = f(x)$$

uniformly for $x \in \mathbb{R}$.

Solution 2. Since f is continuous on the compact set $[0,1]$, it is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon/(3M)$. For any $x \in \mathbb{R}$, let $F(x,t) = f(x-t) - f(x)$, the triangle inequality gives

$$\begin{aligned}\left\| \int_0^1 \mathrm{d}t\, f(x-t)\phi_N(t) - f(x) \right\|_\infty &= \left\| \int_0^1 \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty \\ \leq \left\| \int_0^\delta \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty + \left\| \int_\delta^{1-\delta} \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty &+ \left\| \int_{1-\delta}^1 \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty .\end{aligned}$$

For the first and third integrals, since $|F(x,t)| < \varepsilon/(3M)$ for $|t| < \delta$, which for $t \in \mathbb{R}/\mathbb{Z}$ is equivalent to $t < \delta$ and $t > 1-\delta$, we have

$$\left\| \int_0^\delta \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty < \frac{\varepsilon}{3M} \int_0^\delta |\phi_N(t)|\, dt \leq \frac{\varepsilon}{3},$$

and

$$\left\| \int_{1-\delta}^1 \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty <\leq \frac{\varepsilon}{3M} \int_{1-\delta}^1 |\phi_N(t)|\, dt \leq \frac{\varepsilon}{3}.$$

Since f is uniformly continuous, it is also bounded, so there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$. By assumption, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, we have

$$\int_\delta^{1-\delta} |\phi_N(t)|\, dt < \frac{\varepsilon}{6B}$$

Thus, for the second integral, we have

$$\left\| \int_\delta^{1-\delta} \mathrm{d}t\, F(x,t)\phi_N(t) \right\|_\infty \leq 2B \int_\delta^{1-\delta} |\phi_N(t)|\, dt < 2B \cdot \frac{\varepsilon}{6B} < \frac{\varepsilon}{3}.$$

Therefore, given any $\varepsilon > 0$, for all $x \in \mathbb{R}$ and $N \geq N_0$, we have

$$\left\| \int_0^1 \mathrm{d}t\, f(x-t)\phi_N(t) - f(x) \right\|_\infty < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and hence on $\mathbb{R}$, we have

$$\int_0^1 \mathrm{d}t\, f(x-t)\phi_N(t) \rightrightarrows f(x).$$

Exercise 3 (**Exercise 5.2.3**, 15 points)**.** If $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is a non-zero function, show that

$$0 < \|f\|_2 \;\leq\; \|f\|_\infty .$$

Conversely, if $0 < A \leq B$ are real numbers, show that there exists a non-zero function $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ such that

$$\|f\|_2 = A \qquad \text{and} \qquad \|f\|_\infty = B.$$

(Hint: let g be a non-constant non-negative real-valued function in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and consider functions of the form $f = (c + dg)^{1/2}$ for some constant real numbers $c, d > 0$.)

Solution 3. If f is nonzero, by the definition of the norm we must have $\|f\|_2 > 0$. For each $x \in \mathbb{R}/\mathbb{Z}$, we have $f(x) \leq \|f\|_\infty$. Therefore,

$$\|f\|_2^2 = \int_0^1 \mathrm{d}t\, |f(t)|^2 \leq \int_0^1 \mathrm{d}t\, \|f\|_\infty^2 = \|f\|_\infty^2.$$

Conversely, suppose $0 < A \leq B$ are real numbers. If $A = B$, then let $f = A$ be the constant function and we are done. If $A < B$, let $g(x) = \sin^2(2\pi x) \leq 1$, then $g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is non-constant and non-negative. Consider the function $f(x) = (c + dg(x))^{1/2}$, where $c, d > 0$ are constants to be determined. We have

$$\|f\|_\infty = \max_{x \in \mathbb{R}/\mathbb{Z}} (c + dg(x))^{1/2} = (c + d)^{1/2},$$

and

$$\|f\|_2^2 = \int_0^1 \mathrm{d}t\, (c + dg(t)) = c + d \int_0^1 \mathrm{d}t\, \sin^2(2\pi t) = c + \frac{d}{2}.$$

Thus, we want to solve for c, d such that $(c+d)^{1/2} = B$ and $(c + \frac{d}{2})^{1/2} = A$. The solution is $c = 2A^2 - B^2$, $d = 2\left(B^2 - A^2\right)$, and hence the function

$$f(x) = \left(2A^2 - B^2 + 2\left(B^2 - A^2\right)\sin^2(2\pi x)\right)^{1/2}$$

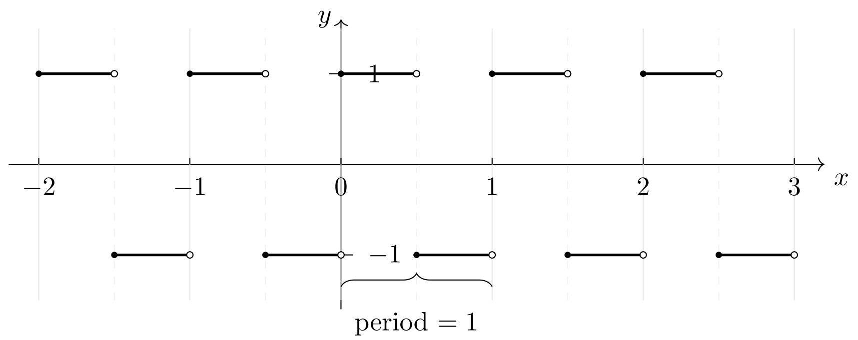
works.

Exercise 4 (15 points)**.** A square wave function is a $\mathbb{Z}$-periodic function defined by

$$f(x) = \begin{cases} 1, & x \in [k,\, k + \frac{1}{2}), \\ -1, & x \in [k + \frac{1}{2},\, k+1), \end{cases} \qquad k \in \mathbb{Z}.$$

Thus f alternates between 1 and -1 on each half-interval, repeating the same pattern on every interval of length 1.

Find a sequence of continuous periodic functions which converges in L^2 to the square wave function.



Solution 4. The square wave function has a discontinuity at $x = k + \frac{1}{2}$. Thus, we can do the following approximation of it. Let

$$f(x) = \begin{cases} 1, & x \in \left[k, k + \dfrac{1}{2} - \dfrac{1}{n}\right), \\ -n\left(x - k - \dfrac{1}{2}\right), & x \in \left[k + \dfrac{1}{2} - \dfrac{1}{n}, k + \dfrac{1}{2}\right), \\ -1, & x \in \left[k + \dfrac{1}{2}, k + 1 - \dfrac{1}{n}\right), \\ 2n(x - k - 1) + 1, & x \in \left[k + 1 - \dfrac{1}{n}, k + 1\right), \end{cases}$$

Then, f is periodic since $f(x+1) = f(x)$ by construction, and f is continuous since the limits of f at $x = k + \frac{1}{2} \pm \frac{1}{n}$ exists and are equal to ± 1. To check the L^2-convergence of f_n to f, by periodicity we focus on the intervals $I_n^{(1)} = [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$ and $I_n^{(2)} = [1 - \frac{1}{n}, 1]$, since $f_n = f$ on $\left[0, \frac{1}{2} - \frac{1}{n}\right] \cup \left[\frac{1}{2}, 1 - \frac{1}{n}\right]$. We have $|I_n^{(1)}| = |I_n^{(2)}| = \frac{1}{n}$, and for any $x \in I_n^{(i)}$, $i = 1, 2$, we have $|f_n(x) - f(x)| \leq 2$. Therefore, for all $\varepsilon > 0$, take $N = 8/\varepsilon$, and we have

$$\|f_n(x) - f(x)\|_2^2 = \int_{I_n^{(1)} \cup I_n^{(2)}} \mathrm{d}t \, |f_n(t) - f(t)|^2 \leq \int_{I_n^{(1)} \cup I_n^{(2)}} \mathrm{d}t \, 4 = \frac{8}{n} < \varepsilon,$$

whenever $n > N$. Thus, $f_n \to f$ in the L^2 metric.

Exercise 5 (15 points)**.**

(a) Evaluate

$$S_n(\theta) = \sum_{k=1}^{n} \sin(k\theta).$$

(b) Show that

$$|S_n(\theta)| \leq \pi\varepsilon^{-1} \qquad \text{on } [\varepsilon, \, 2\pi - \varepsilon] \text{ for all } n \geq 1.$$

Solution 5.

(a) Recall that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. Thus, by the geometric series formula, we have

$$\begin{aligned}
S_n(\theta) = \sum_{k=1}^{n} \sin(k\theta) &= \sum_{k=1}^{n} \frac{e^{ik\theta} - e^{-ik\theta}}{2i} \\
&= \frac{1}{2i} \left(\sum_{k=1}^{n} e^{ik\theta} - \sum_{k=1}^{n} e^{-ik\theta} \right) \\
&= \frac{1}{2i} \left(e^{i\theta} \frac{1 - e^{in\theta}}{1 - e^{i\theta}} - e^{-i\theta} \frac{1 - e^{-in\theta}}{1 - e^{-i\theta}} \right) \\
&= \frac{1}{2i} \left(e^{i(n+1)\theta/2} \frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} - e^{-i(n+1)\theta/2} \frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \right) \\
&= \left(\frac{e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}}{2i} \right) \left(\frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \right) \\
&= \frac{\sin\left(\frac{(n+1)\theta}{2}\right) \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

(b) Let $\theta \in [\varepsilon, 2\pi - \varepsilon]$. Then, since $\sin x$ is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$, we have

$$\left| \sin\left(\frac{\theta}{2}\right) \right| \geq \left| \sin\left(\frac{\varepsilon}{2}\right) \right|.$$

On $[0, \frac{\pi}{2}]$, since $\sin x$ passes through $\left(\frac{\pi}{2}, 1\right)$, and is concave because $(\sin x)'' = -\sin x < 0$, we have $\sin x \geq \frac{2}{\pi} x$. Thus, by part (a), we have

$$|S_n(\theta)| = \left| \frac{\sin\left(\frac{(n+1)\theta}{2}\right) \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right| \leq \frac{1}{\left|\sin\left(\frac{\varepsilon}{2}\right)\right|} \leq \frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|} \leq \frac{\pi}{\varepsilon}.$$

Remark. This implies that $S_n(\theta)$ is uniformly bounded for all n on any compact subset of the interval $(0, 2\pi)$.

Exercise 6 (15 points). Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$. We define their periodic convolution $f * g : \mathbb{R} \to \mathbb{R}$ by

$$(f * g)(x) := \int_0^1 f(y)\, g(x-y)\, dy.$$

Prove that $(f * g)$ is smooth whenever f is smooth. (Remark: A function is called smooth if it has derivatives of all orders.)

Solution 6. First consider $z = x - y$, then let $h = f * g$, we have

$$h(x) = \int_x^{x+1} \mathrm{d}z\, f(x-z)g(z) = \int_0^1 \mathrm{d}z\, f(x-z)g(z),$$

since both f and g are periodic with period 1. For each fixed $y \in [0,1]$, by the Mean Value Theorem, there exists $\xi_t \in (0,t)$ such that

$$\frac{h(x+t) - h(x)}{t} = \int_0^1 \mathrm{d}z\, g(z) \frac{f(x+t-z) - f(x-z)}{t} = \int_0^1 \mathrm{d}z\, g(z) f'(x - z + \xi_t).$$

Since $x - z \in [-1, 1]$, we have $x - z + \xi_t \in [-1-t, 1+t]$. Since f is smooth, f' is continuous on the compact set $[-1-t, 1+t]$, so f' is uniformly continuous there. Thus, for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $|(x - z + \xi_t) - (x - z)| = |\xi_t| < t < \delta$, we have

$$|f'(x - z + \xi_t) - f'(x - z)| < \varepsilon.$$

Since g is continuous on the compact set $[0,1]$, it is bounded, so there exists $M > 0$ such that $|g(z)| \leq M$ for all $z \in [0,1]$. Therefore, for all $t < \delta$, we have

$$\left| \frac{h(x+t) - h(x)}{t} - \int_0^1 \mathrm{d}z\, g(z) f'(x-z) \right| \leq \int_0^1 \mathrm{d}z\, |g(z)|\, |f'(x - z + \xi_t) - f'(x-z)| \leq M\varepsilon,$$

and hence the first derivative exists:

$$h'(x) = \lim_{t \to 0} \frac{h(x+t) - h(x)}{t} = \int_0^1 \mathrm{d}z\, g(z) f'(x-z).$$

We can show the higher-order derivatives similarly by induction. The base case $n = 1$ is done above. Suppose the n-th derivative exists and is given by

$$h^{(n)}(x) = \int_0^1 \mathrm{d}z\, g(z) f^{(n)}(x-z).$$

Then, for the $(n+1)$-th derivative, we have

$$\begin{aligned} \frac{h^{(n)}(x+t) - h^{(n)}(x)}{t} &= \int_0^1 \mathrm{d}z\, g(z) \frac{f^{(n)}(x+t-z) - f^{(n)}(x-z)}{t} \\ &= \int_0^1 \mathrm{d}z\, g(z) f^{(n+1)}(x - z + \xi_t), \end{aligned}$$

for some $\xi_t \in (0,t)$. By the same argument as above, we may switch the order of limit and integration, and thus

$$h^{(n+1)}(x) = \int_0^1 \mathrm{d}z\, g(z) f^{(n+1)}(x-z).$$

Therefore, by induction, h has derivatives of all orders, so h is smooth.