Math 2213 Introduction to Analysis I

Homework 12 Due December 12 (Friday), 2025

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Exercise 1 (**Exercise 5.4.1**, 20 points). Show that if $f : \mathbb{R} \to \mathbb{C}$ is both compactly supported and $\mathbb{Z}$–periodic, then it is identically zero. Hint: A function $f : \mathbb{R} \to \mathbb{C}$ is said to be compactly supported if the set

$$\operatorname{supp}(f) := \overline{\{ x \in \mathbb{R} : f(x) \neq 0 \}}$$

is a compact subset of $\mathbb{R}$. Equivalently, f is compactly supported if there exists a bounded closed interval $[a, b] \subset \mathbb{R}$ such that

$$f(x) = 0 \qquad \text{whenever } x \notin [a, b].$$

Solution 1.

Exercise 2 (**Exercise 5.5.1**, 20 points). Let f be a function in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and define the trigonometric Fourier coefficients a_n, b_n for $n = 0, 1, 2, \ldots$ by

$$a_n := 2\int_0^1 f(x)\cos(2\pi nx)\,dx, \qquad b_n := 2\int_0^1 f(x)\sin(2\pi nx)\,dx.$$

(a) Show that the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(2\pi nx) + b_n\sin(2\pi nx)\big)$$

converges to f in the L^2-metric.

(b) Show that if $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=1}^{\infty}|b_n|$ are absolutely convergent, then the above series actually converges uniformly to f (and not just in L^2).

Solution 2.

(a) By the Fourier Theorem (Theorem 5.5.1), for any $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, we have

$$\lim_{N\to\infty}\left\| f - \sum_{n=-N}^{N}\hat{f}(n)e_n \right\|_2 = 0.$$

That is, the Fourier series $F_N = \sum_{n=-N}^{N}\hat{f}(n)e_n$ converges to f in the L^2-metric. We have $e_n = e^{2\pi inx} = \cos 2\pi nx + i\sin 2\pi nx$.

$$\begin{aligned}
F_N = \sum_{n=-N}^{N}\hat{f}(n)e_n &= \hat{f}(0) + \sum_{n=1}^{N}\left(\hat{f}(n)e_n + \hat{f}(-n)e_{-n}\right)\\
&= \hat{f}(0) + \sum_{n=1}^{N}\left(\hat{f}(n)(\cos 2\pi nx + i\sin 2\pi nx) + \hat{f}(-n)(\cos 2\pi nx - i\sin 2\pi nx)\right)\\
&= \hat{f}(0) + \sum_{n=1}^{N}\left((\hat{f}(n) + \hat{f}(-n))\cos 2\pi nx + i(\hat{f}(n) - \hat{f}(-n))\sin 2\pi nx\right).
\end{aligned}$$

Finally note that the given series is exactly F_N:

$$\begin{aligned}
\hat{f}(n)+\hat{f}(-n) &= \int_{[0,1]} \mathrm{d}x\, f(x)\left(e^{2\pi inx}+e^{-2\pi nx}\right)\\
&= 2\int_{[0,1]} \mathrm{d}x\, f(x)\cos 2\pi nx = a_n, \quad n\geq 2,\\
\hat{f}(0) &= \int_{[0,1]} \mathrm{d}x\, f(x) = \frac{a_0}{2},\\
i(\hat{f}(n)-\hat{f}(-n)) &= i\int_{[0,1]} \mathrm{d}x\, f(x)\left(e^{2\pi inx}-e^{-2\pi nx}\right)\\
&= -2\int_{[0,1]} \mathrm{d}x\, f(x)\sin 2\pi nx = b_n, \quad n\geq 1.
\end{aligned}$$

(b) Theorem 5.5.3 states that for $f\in C(\mathbb{R}/\mathbb{Z},\mathbb{C})$, if $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|<\infty$, then the Fourier series converges uniformly to f. I.e.

$$\lim_{N\to\infty}\left\| f-\sum_{n=-N}^{N}\hat{f}(n)e_n\right\|_\infty = 0.$$

Suppose (a_n) and (b_n) converge absolutely, then

$$S_n=\sum_{k=1}^{n}|a_k|, \quad T_n=\sum_{k=1}^{n}|b_k|, \quad n=1,2,\dots$$

$$\sum_{n=-N}^{N}|\hat{f}(n)| = \sum_{n=-N}^{N}\left|\int_{[0,1]}\mathrm{d}x\, f(x)e^{-2\pi inx}\right| \leq \sum_{n=-N}^{N}\int_{[0,1]}\mathrm{d}x\,|f(x)||e^{-2\pi inx}| = (2N+1)\|f\|_\infty<\infty.$$

$$\sum_{n=-N}^{N}|\hat{f}(n)e_n| = |\hat{f}(0)|+\sum_{n=1}^{N}\left(|\hat{f}(n)|+|\hat{f}(-n)|\right) = \frac{|a_0|}{2}+\sum_{n=1}^{N}\left(\frac{|a_n|}{2}+\frac{|b_n|}{2}\right).$$

Exercise 3 (**Exercise 5.5.2**, 20 points). Let $f(x)$ be the function defined by $f(x)=(1-2x)^2$ when $x\in[0,1]$, and extended to be $\mathbf{Z}$–periodic on $\mathbf{R}$.

(a) Using Exercise 5.5.1, show that the series

$$\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\cos(2\pi nx)$$

converges uniformly to f. *Hint: You may use the fact that*

$$\int_0^1 x\, e^{-2\pi inx}\,dx = -\frac{1}{2\pi in}, \qquad (n\neq 0),$$

$$\int_0^1 x^2\, e^{-2\pi inx}\,dx = -\frac{1}{2\pi in}+\frac{2}{(2\pi n)^2}, \qquad (n\neq 0).$$

(b) Conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

(c) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Hint: expand the cosines in terms of exponentials and use Plancherel's theorem.

Solution 3.

(a) Following Exercise 5.5.1, we compute the Fourier coefficients of f. For $n \geq 1$, we have

$$\int_0^1 \mathrm{d}x\, x e^{-2\pi i n x} = \frac{i}{2\pi n} = \int_0^1 \mathrm{d}x\, x \cos 2\pi n x - i \int_0^1 \mathrm{d}x\, x \sin 2\pi n x,$$

$$\int_0^1 \mathrm{d}x\, x^2 e^{-2\pi i n x} = \frac{1}{2\pi^2 n^2} - \frac{i}{2\pi n} = \int_0^1 \mathrm{d}x\, x^2 \cos 2\pi n x - i \int_0^1 \mathrm{d}x\, x^2 \sin 2\pi n x.$$

Then,

$$\begin{aligned}
a_n &= 2 \int_0^1 (1-2x)^2 \cos(2\pi n x)\, dx \\
&= 2 \int_0^1 (1 - 4x + 4x^2) \cos(2\pi n x)\, dx \\
&= -8(0) + 8\left(\frac{1}{2\pi^2 n^2}\right) = \frac{4}{\pi^2 n^2}, \quad n \geq 2, \\
a_0 &= 2 \int_0^1 (1-2x)^2\, dx = \frac{2}{3}, \\
b_n &= 2 \int_0^1 (1-2x)^2 \sin(2\pi n x)\, dx \\
&= 2 \int_0^1 (1 - 4x + 4x^2) \sin(2\pi n x)\, dx \\
&= 0 - 8\left(\frac{1}{2\pi n}\right) + 8(0) = 0, \quad n \geq 1.
\end{aligned}$$

Hence, the Fourier series of f is

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x).$$

Since $\sum_{n=1}^{\infty} \left| \frac{4}{\pi^2 n^2} \right|$ converges, by Exercise 5.5.1(b), the series converges uniformly to f.

(b) Plugging in $x = 0$ gives

$$f(0) = 1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(c) We can write

$$F_N = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} (e^{2\pi i n x} + e^{-2\pi i n x}),$$

and so the complex Fourier coefficients are

$$\hat{f}(0) = \frac{1}{3}, \quad \hat{f}(n) = \frac{2}{\pi^2 n^2}, \quad n \geq 1.$$

By Plancherel's theorem, we have

$$\int_0^1 \mathrm{d}x\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{9} + \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

On the other hand, we have

$$\int_0^1 \mathrm{d}x\, (1-2x)^4 = \int_0^1 \mathrm{d}x\, (1 - 8x + 24x^2 - 32x^3 + 16x^4) = \frac{1}{5}.$$

Hence,

$$\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Exercise 4 (**Exercise 5.5.3**, 20 points). If $f \in C(\mathbf{R}/\mathbb{Z}; \mathbb{C})$ and P is a trigonometric polynomial, show that

$$\widehat{f * P}(n) = \widehat{f}(n)\, c_n = \widehat{f}(n)\, \widehat{P}(n)$$

for all integers n, where c_n are the Fourier coefficients of P. More generally, if $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, show that

$$\widehat{f * g}(n) = \widehat{f}(n)\, \widehat{g}(n) \quad \text{for all } n \in \mathbf{Z}.$$

Solution 4.

Exercise 5 (**Exercise 5.5.4**, 20 points). Let $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ be differentiable, and assume its derivative f' is also continuous. Show that

$$\sum_{n=-\infty}^{\infty} |n\, \widehat{f}(n)|^2 < \infty$$

and that the Fourier coefficients of f' satisfy

$$\widehat{f'}(n) = 2\pi i n\, \widehat{f}(n) \qquad \text{for all } n \in \mathbf{Z}.$$

Solution 5.

You can do the following problems to practice.
You don't have to submit the following problems.

Exercise 6 (**Exercise 5.5.5**, Optional). Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Prove the Parseval identity

$$\Re \int_0^1 f(x)\overline{g(x)}\, dx = \Re \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \overline{\widehat{g}(n)}.$$

Hint: apply the Plancherel theorem to $f+g$ and $f-g$, and subtract the two. Then conclude that the real parts can be removed, i.e.

$$\int_0^1 f(x)\overline{g(x)}\, dx = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, \overline{\widehat{g}(n)}.$$

Hint: apply the first identity with f replaced by if.

Solution 6.

Exercise 7 (**Exercise 5.5.6**, Optional). In this exercise we develop Fourier series for functions of an arbitrary period $L > 0$. Let $L > 0$ and let $f : \mathbb{R} \to \mathbb{C}$ be a continuous L-periodic function. For each integer n define

$$c_n := \frac{1}{L}\int_0^L f(x)\, e^{-2\pi i n x/L}\, dx.$$

(a) Show that the series

$$\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n x/L}$$

converges to f in L^2-metric. More precisely, prove that

$$\lim_{N\to\infty} \int_0^L \Big| f(x) - \sum_{n=-N}^{N} c_n\, e^{2\pi i n x/L} \Big|^2 dx = 0.$$

Hint: apply the Fourier theorem to the function $f(Lx)$.

(b) If the series $\sum_{n=-\infty}^{\infty} |c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n x/L}$$

converges uniformly to f.

(c) Show that

$$\frac{1}{L}\int_0^L |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Hint: apply the Plancherel theorem to the function $f(Lx)$.

Solution 7.