Math 2213 Introduction to Analysis I

Homework 2 Due September 17 (Thursday), 2025

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Definition 1 (metric). A function $d : X \times X \to [0, \infty)$ is called a metric on X if, for all $x, y, z \in X$, the following properties hold:

(i) For any $x \in X$, we have $d(x, x) = 0$.

(ii) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(iii) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(iv) (Triangle Inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2 (Interior, exterior, boundary points). Let (X, d) be a metric space, let $E \subseteq X$, and let $x_0 \in X$. We say that x_0 is an interior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an exterior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \varnothing$. We say that x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.

Definition 3 (Closure). Let (X, d) be a metric space, let $E \subseteq X$, and let $x_0 \in X$. We say that x_0 is an adherent point of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E; i.e., $B(x_0, r) \cap E \neq \varnothing$. The set of all adherent points of E is called the closure of E and is denoted $\overline{E}$.

Definition 4 (Open and closed sets). Let (X, d) be a metric space, and let E be a subset of X. We say that E is closed if it contains all of its boundary points, i.e., $\partial E \subseteq E$. We say that E is open if it contains none of its boundary points, i.e., $\partial E \cap E = \varnothing$. If E contains some of its boundary points but not others, then it is neither open nor closed.

Problem 1 (11 pts). If (X, d) is a metric space, define

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}. \tag{1}$$

Prove that d' is also a metric on X. Note that $0 \leq d'(x, y) < 1$ for all $x, y \in X$.

Solution 1. We shall verify that d' satisfies the definition of a metric (1).

(i) For any $x \in X$, we have $d'(x, x) = d(x, x)/(1 + d(x, x)) = 0$.

(ii) For any distnct $x, y \in X$, $d'(x, y) = d(x, y)/(1 + d(x, y)) > 0$ since $d(x, y) > 0$.

(iii) For any $x, y \in X$, $d'(x, y) = d(x, y)/(1 + d(x, y)) = d(y, x)/(1 + d(y, x)) = d'(y, x)$ by the symmetry of d.

(iv) For any $x, y, z \in X$, we have

$$\begin{aligned} d'(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= d'(x, y) + d'(y, z). \end{aligned} \tag{2}$$

The first inequality follows from the triangle inequality of d:

$$\begin{aligned} d'(x, z) &= \frac{d(x, z)}{1 + d(x, z)} = \left(1 + \frac{1}{d(x, z)}\right)^{-1} \\ &\leq \left(1 + \frac{1}{d(x, y) + d(y, z)}\right)^{-1} \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)}. \end{aligned} \tag{3}$$

Problem 2 (Exercise 1.2.4 (12 pts)). Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball

$$B \equiv B(x_0, r) = \{x \in X : d(x, x_0) < r\}, \tag{4}$$

and let C be the closed ball

$$C \equiv \{x \in X : d(x, x_0) \leq r\}. \tag{5}$$

(a) Show that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B} \neq C$.

Solution 2.

(a) Following definition (3), let $x \in \overline{B}$, then $B(x, r') \cap B \neq \varnothing$ for any $r' > 0$. Thus, there exists some $y \in B(x, r') \cap B(x_0, r)$, y satisfies $d(x, y) < r'$ and $d(y, x_0) < r$. By the triangle inequality, $d(x, x_0) \leq d(x, y) + d(y, x_0) < r' + r$ for any $r' > 0$, so $d(x, x_0) \leq r$. Therefore, $x \in C$, and $\overline{B} \subseteq C$.

(b) Let d be the discrete metric and X be any set with $|X| \geq 2$. Then for any $x \in X$ and $r = 1$, $B_{(X,d)}(x, r) = \{x\}$, $\overline{B} = \{x\}$. However, the closed ball $C = \overline{B}(x_0, r)$ is all of X. We may conclude that the closure of an open ball is not always the corresponding closed ball, i.e. $\overline{B(x, r)} \neq \overline{B}(x, r)$.

Problem 3 (21 pts). Two metrics d_1 and d_2 on a set X are said to be Lipschitz equivalent if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y) \quad \text{for all } x, y \in X. \tag{6}$$

Let $E \subset X$.

(a) Prove that E is open in (X, d_1) if and only if E is open in (X, d_2).

(b) Prove that E is closed in (X, d_1) if and only if E is closed in (X, d_2).

(c) Two metrics d_1 and d_2 on a set X are said to be topologically equivalent if they induce the same topology on X. That is, a set $U \subset X$ is open in (X, d_1) if and only if it is open in (X, d_2). Give examples of topologically equivalent metrics that are not Lipschitz equivalent.

Solution 3.

(a) Suppose E is open in (X, d_1), then by Proposition 1.2.15 (a), there exists $r > 0$ such that $B_{d_1}(x, r) \subseteq E$ for any $x \in E$. By the left inequality of equation (6), we have

$$d_2(x, y) \leq \frac{1}{C_1} d_1(x, y) < \frac{r}{C_1}, \tag{7}$$

Thus, $x \in B_{(X,d_2)}(x, r/C_1) \subseteq B_{(X,d_2)}(x, r) \subseteq E$ and E is open in (X, d_2). Conversely, suppose E is open in (X, d_2), then there exists $r > 0$ such that $B_{d_2}(x, r) \subseteq E$ for any $x \in E$. By the right inequality of equation (6), we have

$$d_1(x, y) \leq C_2 d_2(x, y) < C_2 r, \tag{8}$$

Thus, $x \in B_{(X,d_1)}(x, C_2 r) \subseteq B_{(X,d_1)}(x, r) \subseteq E$ and E is open in (X, d_1).

(b) By Proposition 1.2.15 (e), E is open if and only if $E^c \equiv X/E$ is closed. Thus, by part (a), E is closed in (X, d_1) if and only if E^c is open in (X, d_1) if and only if E^c is open in (X, d_2) if and only if E is closed in (X, d_2).

(c) Consider the metrics $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\tan x - \tan y|$ on $S = (0, \pi/2) \subseteq \mathbb{R}$. Let $U \subseteq S$ be d_1-open, then for any $x \in U$, there exists $r_x > 0$ such that $B_{(S,d_1)}(x, r_x) \subseteq U$. Then

$$|\tan y - \tan x| = \frac{|\tan(y - x)|}{1 + \tan x \tan y} \leq |\tan(y - x)| = \tan|y - x| < \tan r_x, \tag{9}$$

so $B_{(S,d_2)}(x, \tan r_x) \subseteq U$. Conversely, suppose $U \in S$ is d_2-open, then there exists $r_x > 0$ such that $B_{(S,d_2)}(x, r_x) \subseteq U$. Then

$$|y - x| = |\arctan(\tan y) - \arctan(\tan x)| = \left| \int_{\tan x}^{\tan y} \frac{1}{1 + t^2} dt \right| \leq |\tan y - \tan x| < r_x, \tag{10}$$

so $B_{(S,d_1)}(x, r_x) \subseteq U$. Therefore, d_1 and d_2 are topologically equivalent. However, d_1 is bounded on S while d_2 is unbounded, so they cannot be Lipschitz equivalent.

Problem 4 (15 pts). Let $\mathcal{M}_n = M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Define a function on $\mathcal{M}_n \times \mathcal{M}_n$ by

$$\rho(A, B) = \operatorname{rank}(A - B). \tag{11}$$

Then ρ is a metric on $\mathcal{M}_n$ and it is topologically equivalent to the discrete metric on $\mathcal{M}_n$.

Solution 4. First we verify that ρ is a metric on $\mathcal{M}_n$ by verifying the four properties of definition (1).

(i) $\rho(A, A) = 0$ since the rank of the zero matrix is zero.

(ii) For any distinct $A, B \in \mathcal{M}_n$, we have $\rho(A, B) = \text{rank}(A - B) > 0$ since $A - B$ is a non-zero matrix and the rank of a non-zero matrix is positive.

(iii) For any $A, B \in \mathcal{M}_n$, we have $\rho(A, B) = \text{rank}(A - B) = \text{rank}((-1)(B - A)) = \text{rank}(B - A) = \rho(B, A)$, since multiplication by a nonzero scalar does not change the rank.

(iv) For any $X, Y \in \mathcal{M}_n$, let $\{e_i\}$ and $\{f_j\}$ be the bases for the columns of X and Y, respectively, then $\{e_i\} \cup \{f_j\}$ spans the columns of $X + Y$. Hence $\text{rank}(X + Y) \leq |\{e_i\} \cup \{f_j\}| \leq |\{e_i\}| + |\{f_j\}| = \text{rank}(X) + \text{rank}(Y)$. Therefore, for any $A, B, C \in \mathcal{M}_n$, we have $\rho(A, C) = \text{rank}(A - C) = \text{rank}((A - B) + (B - C)) \leq \text{rank}(A - B) + \text{rank}(B - C) = \rho(A, B) + \rho(B, C)$.

Denote the discrete metric by d. Any $U \subseteq \mathcal{M}_n$ is d-open in $\mathcal{M}_n$, since for any $A \in U$, we have $B_d(A, 1) = \{A\} \subseteq U$. Conversely, $\rho(A, B) = \text{rank}(A - B) \geq 1$ if and only if $A \neq B$, so $B_\rho(A, 1) = \{A\}$. Thus, any $U \subseteq \mathcal{M}_n$ is ρ-open in $\mathcal{M}_n$. All subsets are d- and ρ-open, so a subset is open in $(\mathcal{M}_n, d)$ if and only if it is open in $(\mathcal{M}_n, \rho)$. Therefore, d and ρ are topologically equivalent.

Problem 5 (20 pts)**.** Let E be a subset of a metric space (X, d). Prove the following:

(a) The boundary of E is a closed set.

(b) $\partial E = \overline{E} \cap \overline{X \setminus E}$

(c) If E is clopen (closed and open), what is ∂E?

(d) Give an example of $S \subset \mathbb{R}$ such that $\partial(\partial S) \neq \varnothing$, and infer that **"the boundary of the boundary $\partial \circ \partial$ is not always zero."**

Solution 5.

(a) By the result of (b), ∂E is closed since it is the intersection of two closed sets by Proposition 1.2.15.

(b) Suppose $x \in \partial E$, then x is not interior to E, so $B(x, r) \cap X \backslash E \neq \varnothing$ for all $r > 0$, hence $x \in \overline{E}$; x is not exterior to E, so $B(x, r) \cap E \neq \varnothing$ for all $r > 0$, hence $x \in \overline{X \backslash E}$. Therefore, $\partial E \subseteq \overline{E} \cap \overline{X \backslash E}$. Conversely, suppose $x \in \overline{E} \cap \overline{X \backslash E}$, then for all $r > 0$, $B(x, r) \cap E \neq \varnothing$ and $B(x, r) \cap X \backslash E \neq \varnothing$, so x is neither interior nor exterior to E, hence $x \in \partial E$. Therefore, $\partial E = \overline{E} \cap \overline{X \backslash E}$.

(c) If E is clopen, then by definition (4) $\partial E \subseteq E$ and $\partial E \cap E = \varnothing$. Thus $\partial E = \varnothing$.

(d) Consider the set $S = \{x \in \mathbb{Q} \mid 2 \leq x \leq 4\} \subset \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\partial S = [2, 4] \subseteq \mathbb{R}$. Thus, $\partial(\partial S) = \{2, 4\} \neq \varnothing$, giving an example where $\partial \circ \partial$ is not zero.

Problem 6. Let (X, d) be a metric space. If subsets satisfy $A \subseteq S \subseteq \overline{A}^S$, where $\overline{A}^S$ denotes the closure of A with respect to the subspace metric on S, then A is said to be *dense in* S. Recall that the closure of A in the subspace $(S, d|_{S\times S})$ is defined by

$$\overline{A}^S \equiv \{s \in S : \forall r > 0, B_S(s, r) \cap A \neq \varnothing\},$$

where $B_S(s, r) = B_X(s, r) \cap S$ is the open ball in S relative to X. Equivalently, A is dense in S if for every $s \in S$ and $r > 0$ one has

$$B_X(s, r) \cap S \cap A \neq \emptyset.$$

(a) Suppose $A \subseteq S \subseteq T$. If A is dense in S and S is dense in T, prove that A is dense in T. Equivalently,

$$\overline{A}^S = S \quad \text{and} \quad \overline{S}^T = T \quad \Rightarrow \quad \overline{A}^T = T,$$

where $\overline{\cdot}^Y$ denotes closure in the subspace Y.

(b) If A is dense in S and B is open in S, prove that $B \subseteq \overline{A \cap B}^S$.

Note: B is open in S iff $B = V \cap S$ for some open $V \subseteq X$, equivalently, for every $b \in B$ there exists $r > 0$ such that

$$B_S(b, r) = B_X(b, r) \cap S \subseteq B.$$

(c) If A and B are both dense in S and B is open in S, prove that $A \cap B$ is dense in S.

Solution 6.

(a) Suppose A is dense in S and S is dense in T, then for any $s \in S$, $t \in T$, and r_A, $r_S > 0$, we have $B_X(s, r_S) \cap S \cap A \neq \varnothing$ and $B_X(t, r_T) \cap T \cap S \neq \varnothing$. For any $t \in T$, $r > 0$, there exists some $s \in S$ such that $d(t, s) < r/2$, and there exists some $a \in A$ such that $d(s, a) < r/2$. By the triangle inequality, $d(t, a) \leq d(t, s) + d(s, a) < r$, so $a \in B_X(t, r) \cap T \cap A$. Therefore, A is dense in T.

(b) Suppose A is dense in S and B is open in S. Let $x \in B$, then there exists $r > 0$ such that $B_S(x, r) \subseteq B$. By the density of A in S, since $x \in B \subseteq S$, for any $r' > 0$, $B_X(x, r') \cap S \cap A = B_S(x, r') \cap A \neq \varnothing$. Since $B_S(x, r') \subseteq B$ whenever $r' < r$, we have $\varnothing \neq B_S(x, r) \cap A \subseteq B_S(s, r) \cap A \cap B$ whenever $r' < r$, hence the desired result.

(c) Suppose A and B are dense in S and B is open in S. Then by the left inclusion, $A \cap B \subseteq S$, and by (b), $S \subseteq \overline{B} \subseteq \overline{A \cap B}$. Therefore, $A \cap B \subseteq S \subseteq \overline{A \cap B}$, and $A \cap B$ is dense in S.