Math 2213 Introduction to Analysis I

Homework 3 Due September 25 (Thursday), 2025

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Problem 1 (10 pts)**.** (10 pts) Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set

$$S = \{x^{(n)} : n \geq m\}.$$

Is the converse true?

Solution 1.

Problem 2 (20 pts)**.** The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the <u>formal limit</u>

$$\mathrm{LIM}_{n\to\infty} x_n.$$

We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that this equality relation obeys the reflexive, symmetry, and transitive axioms, i.e. that it is an equivalence relation.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, modulo the above equivalence relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (the limit exists and does not depend on the choice of representatives) and that it satisfies the axioms of a metric. Thus $(\overline{X}, d_{\overline{X}})$ is a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding constant Cauchy sequence $(x, x, x, \dots)$, i.e. with the formal limit $\mathrm{LIM}_{n\to\infty} x$. Show that this is legitimate: for $x, y \in X$,

$$x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y.$$

With this identification, show that

$$d(x, y) = d_{\overline{X}}(x, y),$$

and thus (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ itself. (This explains the choice of notation.)

(f) Finally, show that the formal limit agrees with the actual limit: if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X that converges in X, then

$$\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \quad \text{in } \overline{X}.$$

Solution 2.

(a) We show that $\mathrm{LIM}_{n\to\infty}x_n = \lim_{n\to\infty} x_n$ is an equivalence relation.

(i) Reflexivity: $\lim_{n\to\infty} d(x_n, x_n) = 0$ by definition of a metric.

(ii) Symmetry: $\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d(x_n, y_n) = 0$ by symmetry of a metric.

(iii) Transitivity: Suppose $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. By triangle inequality, we have $\lim_{n\to\infty} d(x_n, z_n) \le \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) = 0$.

(b) Since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy sequences, for all $\epsilon > 0$, there exists $N > 0$ such that $d(x_n, x_m) < \epsilon/2$ and $d(y_n, y_m) < \epsilon/2$ for all $n, m > N$. Then

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m) < \epsilon,$$

hence the sequence $(d(x_n, y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\lim_{n\to\infty} d(x_n, y_n)$ exists. Next, suppose $\mathrm{LIM}_{n\to\infty}x_n = \mathrm{LIM}_{n\to\infty}x'_n$, $\mathrm{LIM}_{n\to\infty}y_n = \mathrm{LIM}_{n\to\infty}y'_n$, then $\lim_{n\to\infty} d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} d(y_n, y'_n) = 0$. By triangle inequality, we have

$$\lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n, x'_n) + \lim_{n\to\infty} d(x'_n, y'_n) + \lim_{n\to\infty} d(y'_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

Similarly, we can show that $\lim_{n\to\infty} d(x'_n, y'_n) \le \lim_{n\to\infty} d(x_n, y_n)$. Hence $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$, and $d_{\overline{X}}$ is well-defined.

Next, we check the metric definition. For clarity we will use the following notation: $\tilde{x} \equiv \mathrm{LIM}_{n\to\infty}x_n$, $\tilde{y} \equiv \mathrm{LIM}_{n\to\infty}y_n$, $\tilde{z} \equiv \mathrm{LIM}_{n\to\infty}z_n \in \overline{X}$.

(i) $d_{\overline{X}}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n) = 0$ if and only if $\tilde{x} = \tilde{y}$. Otherwise $d_{\overline{X}}(\tilde{x}, \tilde{y}) > 0$ by positivity of d.

(ii) $d_{\overline{X}}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = d_{\overline{X}}(\tilde{y}, \tilde{x})$, by symmetry of d.

(iii) $d_{\overline{X}}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = d_{\overline{X}}(\tilde{x}, \tilde{z}) + d_{\overline{X}}(\tilde{z}, \tilde{y})$, by triangle inequality of d and the fact that both $\lim_{n\to\infty} d(x_n, z_n)$ and $\lim_{n\to\infty} d(z_n, y_n)$ exist.

(c) A metric space is complete if every Cauchy sequence converges. Let $\left(\mathrm{LIM}_{n\to\infty}x_n^{(m)}\right)_{m=1}^{\infty}$ be a Cauchy sequence in $\overline{X}$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n^{(m)}, \mathrm{LIM}_{n\to\infty}x_n^{(k)}) < \epsilon$$

whenever $m, k > N$. Hence there exists $M > 0$ such that $d(x_n^{(m)}, x_n^{(k)}) < \epsilon$ for all $n > M$, and $\left(x_n^{(m)}\right)_{m=1}^{\infty}$ is Cauchy in X for some fixed $n > M$. By definition of $d_{\overline{X}}$, we have

$$\lim_{n\to\infty} d(x_n^{(m)}, x_n^{(k)}) < \epsilon.$$

Thus, for each fixed n, $(x_n^{(m)})_{m=1}^{\infty}$ is a Cauchy sequence in X and hence converges to some limit $x_\infty^{(m)} \in \overline{X}$, i.e.

$$\mathrm{LIM}_{n\to\infty}x_n^{(m)} = x_\infty^{(m)} \quad \text{for all } m.$$

For all $\epsilon > 0$, there exists $N > 0$ such that

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n^{(m)}, \mathrm{LIM}_{k\to\infty}x_k^{(k)}) = \lim_{n\to\infty} d\left(x_n^{(k)}, x_n^{(n)}\right) < \epsilon.$$

Hence $\lim_{m\to\infty} \mathrm{LIM}_{n\to\infty}x_n^{(m)} \in \overline{X}$, and $(\overline{X}, \overline{d})$ is complete.

(d) Suppose $x, y \in X$. Then $x = y$ if and only if $d(x, y) = 0$ if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$ for $(x_n)_{n=1}^{\infty} = (x, x, \dots)$ and $(y_n)_{n=1}^{\infty} = (y, y, \dots)$ if and only if $\mathrm{LIM}_{n\to\infty}x_n = \mathrm{LIM}_{n\to\infty}y_n$. Therefore, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

(e) Denote the closure as $\tilde{X}$. Let $x \in \tilde{X}$, then for all $\epsilon > 0$, there exists $y \in X$ such that $d_{\overline{X}}(x, y) < \epsilon$. Since $y \in X$, the Cauchy sequence $(y_n)_{n=1}^{\infty} = (y, y, \dots)$ satisfies $\mathrm{LIM}_{n\to\infty} y_n = y$. Then

$$d_{\overline{X}}(x, y) = \lim_{n\to\infty} d(x_n, y_n) < \epsilon,$$

where x, y here stand for the constant sequences $(x, x, \dots)$ and $(y, y, \dots)$ respectively. Hence $x \in \overline{X}$. Conversely, let $x \in \overline{X}$, then $x = \mathrm{LIM}_{n\to\infty} x_n$ for some Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X. Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m > N$. Take $y = x_{N+1} \in X$, then by definition of $d_{\overline{X}}$, we have

$$d_{\overline{X}}(x, y) = \lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(x_n, x_{N+1}) < \epsilon.$$

Hence $x \in \tilde{X}$. Therefore, $\tilde{X} = \overline{X}$.

(f) Suppose $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X converging in X. Then there exists $x \in X$ such that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ whenever $n > N$. By definition of $d_{\overline{X}}$, we have

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, x) = \lim_{n\to\infty} d(x_n, x) = 0,$$

where x in $d_{\overline{X}}$ stands for the constant sequence $(x, x, \dots)$. Hence $\mathrm{LIM}_{n\to\infty} x_n = x$ in $\overline{X}$.

Problem 3 (20 pts)**.** In the following, all the sets are subsets of a metric space (X, d).

(a) If $\overline{A} \cap \overline{B} = \varnothing$, then

$$\partial(A \cup B) = \partial A \cup \partial B.$$

(b) For a finite family $\{A_i\}_{i=1}^{n} \subseteq X$, show that

$$\mathrm{int}\Big(\bigcap_{i=1}^{n} A_i\Big) = \bigcap_{i=1}^{n} \mathrm{int}(A_i).$$

(c) For an arbitrary (possibly infinite) family $\{A_\alpha\}_{\alpha\in F} \subseteq X$, prove that

$$\mathrm{int}\Big(\bigcap_{\alpha\in F} A_\alpha\Big) \subseteq \bigcap_{\alpha\in F} \mathrm{int}(A_\alpha).$$

(d) Give an example where the inclusion in part (c) is strict (i.e., equality fails).

(e) For any family $\{A_\alpha\}_{\alpha\in F} \subseteq M$, prove that

$$\bigcup_{\alpha\in F} \mathrm{int}(A_\alpha) \subseteq \mathrm{int}\Big(\bigcup_{\alpha\in F} A_\alpha\Big).$$

(f) Give an example of a finite collection F in which equality does not hold in part (e).

Solution 3.

Problem 4 (10 pts)**.** Let (X, d) be a metric space and $Y \subset X$ be an open subset. For any subset $A \subset Y$, show that A is open in Y if and only if it is open in X.

Solution 4.

Problem 5 (20 pts)**.** On the space $(0,1]$, we may consider the topology induced by the metric space $(\mathbb{R}, d)$ defined by $d(x,y) = |x-y|$. Alternatively, we may also define a distance d' on $(0,1]$, given by

$$d'(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x, y \in (0,1].$$

(a) Show that d' is a metric on $(0,1]$

(b) Let $x \in (0,1]$ and $\varepsilon > 0$. Let $B = B_d(x,\varepsilon) = \{y \,|\, |y-x| < \varepsilon\} \cap (0,1]$ be the open ball centered at x of radius ε for the metric d in $(0,1]$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_{d'}(y,\varepsilon') \subseteq B = B_d(x,\varepsilon).$$

(c) Show that an open ball in $((0,1], d')$ is also an open ball in $((0,1], d)$.

(d) Conclude that the metric spaces $((0,1], d)$ and $((0,1], d')$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(e) Is $((0,1], d')$ a complete metric space? How about $((0,1], d)$?

Solution 5.

(a) We show that d' satisifes the definition a metric on $(0,1]$.

  (i) For all $x, y \in \mathbb{R}$, $d'(x,x) = |1/x - 1/x| = 0$.

  (ii) For all distinct $x, y \in \mathbb{R}$, $d'(x,y) > 0$.

  (iii) For all $x, y \in \mathbb{R}$, $d'(x,y) = |1/x - 1/y| = |1/y - 1/x| = d'(y,x)$.

  (iv) For all $x, y, z \in \mathbb{R}$, $d'(x,y) = |1/x-1/y| \leq |1/x-1/z| + |1/z-1/y| = d'(x,z) + d'(z,y)$.

(b) Let

(c) Let $B = B_{((0,1],d')}(x,r)$ be an open ball in $((0,1], d')$. Then for all $y \in B$, we have $d'(x,y) = |1/x - 1/y| < r$. By triangle inequality, we have

$$|x-y| = \left|\frac{xy}{y} - \frac{xy}{x}\right| = |xy| \cdot |\frac{1}{x} - \frac{1}{y}| < |xy|r \leq r.$$

Hence B is also an open ball in $((0,1], d)$.

(d) Conversely to (c), let $S \subseteq (0,1]$ be an open set. We can find an open ball $B = B_{((0,1],d)}(x,r) \subseteq S$. Then for all $y \in S$, we have $d(x,y) = |x-y| < r$. By triangle inequality, we have

$$|\frac{1}{x} - \frac{1}{y}| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{|xy|} < \frac{r}{|xy|} \leq r.$$

Hence B is also an open ball in $((0,1], d')$, and $((0,1], d)$ is topologically equivalent to $((0,1], d')$.

(e) $((0,1], d)$ is not complete since the Cauchy sequence $(1/n)_{n=1}^{\infty}$ does not converge in $(0,1]$. However, $((0,1], d')$ is complete since for any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in $((0,1], d')$, the sequence $(1/x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges to some limit $L \in \mathbb{R}$. Since $x_n \in (0,1]$, we have $1/x_n \geq 1$ for all n, and hence $L \geq 1$. Thus, the sequence $(x_n)_{n=1}^{\infty}$ converges to $1/L \in (0,1]$.

Problem 6 (20 pts)**.**

(a) We say that a family of closed balls

$$\big(\overline{B}(x_n, r_n)\big)_{n\geq 1}$$

is a <u>decreasing sequence of closed balls</u> if the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied. Give an example of a decreasing sequence of closed balls in a complete metric space with empty intersection.

(b) We say that a family of closed balls

$$\big(\overline{B}(x_n, r_n)\big)_{n\geq 1}$$

is a <u>decreasing sequence of closed balls with radii tending to zero</u> if

$$r_n \to 0 \quad \text{as } n \to \infty,$$

and the nesting condition

$$\overline{B}(x_{n+1}, r_{n+1}) \subseteq \overline{B}(x_n, r_n) \quad \text{for all } n \in \mathbb{N}$$

is satisfied.

Show that a metric space (M, d) is complete if and only if every decreasing sequence of closed balls with radii going to zero has a nonempty intersection.

Solution 6.

(a) Consider the metric space $(\mathbb{N}, d)$, where

$$d(m,n) = \begin{cases} 0 & m = n, \\ 1 + \frac{1}{\min\{m,n\}} & m \neq n. \end{cases}$$

This is a metric space since it satisfies the definition of a metric:

(i) For all $m, n \in \mathbb{N}$, we have $d(m,n) \geq 0$ and $d(m,n) = 0$ if and only if $m = n$ by construction.

(ii) For all $m, n \in \mathbb{N}$, we have $d(m,n) = d(n,m)$ by symmetry of $\min(\cdot,\cdot)$.

(iii) For all $m, n, p \in \mathbb{N}$, we have

$$\begin{aligned} d(m,n) = 1 + \frac{1}{\min\{m,n\}} &\leq 1 + \frac{1}{\min\{m,p\}} + 1 + \frac{1}{\min\{p,n\}} \\ &= d(m,p) + d(p,n), \end{aligned}$$

since we can check that the inequality holds for all the cases: $p \leq \min\{m,n\}$, $\min\{m,n\} < p < \max\{m,n\}$, $\max\{m,n\} \leq p$.

Only same point sequences $(x, x, x, \dots)$ are Cauchy sequences in $(\mathbb{N}, d)$, hence they converge in $\mathbb{N}$ and $(\mathbb{N}, d)$ is complete. Take $\big(\overline{B}(n, r_n)\big)_{n\geq 1}$, where $r_n = 1 + \frac{1}{n}$. Then

$$\overline{B}(n+1, r_{n+1}) = [n+1, \infty) \subseteq [n, \infty) = \overline{B}(n, r_n),$$

so nesting property is satisfied. However, the intersection is empty since

$$\bigcap_{n=1}^{\infty} \overline{B}(n, r_n) = \bigcap_{n=1}^{\infty} [n, \infty) = \varnothing.$$

(b) Suppose (M,d) is a complete metric space. Let $\left(\overline{B}(x_n, r_n)\right)_{n\geq 1}$ be a decreasing sequence of closed balls with radii going to zero. Take $x_n \in \overline{B}(x_n, r_n)$ for all n, and for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $r_n < \epsilon/2$ whenever $n > N$. Notice that

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < r_N + r_N < \epsilon,$$

so $(x_n)_{n=1}^{\infty}$ is a convergent Cauchy sequence in (M, d), and thus there exists $x \in M$ such that $x_n \to x$. For all n, since $x_n \in \overline{B}(x_n, r_n)$, we have $d(x_n, x) \leq r_n$, hence $x \in \overline{B}(x_n, r_n)$, and the intersection is non-empty.

Conversely, suppose every decreasing sequence of closed balls with radii going to zero has a nonempty intersection. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in (M, d). Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m > N$. By assumption for all $\epsilon > 0$, $r_n < \epsilon$ whenever $n > N'$ for some $N' \in \mathbb{N}$.

Notice that in (a) the radii do not tend to zero.