Math 2213 Introduction to Analysis I

Homework 4 Due September 26 (Friday), 2025

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October 13, 2025

Problem 1 (16 pts).

(a) Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the ℓ^1 and ℓ^∞ metrics on this space by

$$d_{\ell^1}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sum_{n=0}^{\infty} |a_n - b_n|,$$

$$d_{\ell^\infty}\left((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\right) := \sup_{n\in\mathbb{N}} |a_n - b_n|.$$

Show that these are both metrics on X, but show that there exist sequences

$$x^{(1)}, x^{(2)}, \ldots$$

of elements of X (i.e. sequences of sequences) which are convergent with respect to the d_{ℓ^∞} metric but not with respect to the d_{ℓ^1} metric. Conversely, show that any sequence which converges in the d_{ℓ^1} metric automatically converges in the d_{ℓ^∞} metric.

(b) Let (X, d_{ℓ^1}) be the metric space from part (a). For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in X such that

$$e_j^{(n)} := \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n \neq j. \end{cases}$$

Show that the set

$$\{e^{(n)} : n \in \mathbb{N}\}$$

is a closed and bounded subset of X, but is not compact.

(This is despite the fact that (X, d_{ℓ^1}) is even a complete metric space—a fact which we will not prove here. The problem is not that X is incomplete, but rather that it is "infinite-dimensional," in a sense that we will not discuss here.)

Solution 1.

(a) We will check the metric axioms for both d_{ℓ^1} and d_{ℓ^∞}. For brevity, we shall denote an element of X by (a) instead of $(a_n)_{n=0}^{\infty}$.

(i) For all $(a), (b) \in X$, $d_{\ell_1}((a), (b)) = 0$ whenever $(a) = (b)$, and $d_{\ell_1}((a), (b)) > 0$ whenever $(a) \neq (b)$, since there exists some $i \in \mathbb{N}$ such that $|a_i - b_i| > 0$.

(ii) For all $(a), (b) \in X$, $d_{\ell_1}((a), (b)) = d_{\ell_1}((b), (a))$ since $|a_i - b_i| = |b_i - a_i|$ for all $i \in \mathbb{N}$.

(iii) For all $(a), (b), (c) \in X$, by triangle inequality of real numbers with respect to $|\cdot, \cdot|$, we have

$$d_{\ell_1}((a), (c)) \leq d_{\ell_1}((a), (b)) + d_{\ell_1}((b), (c)).$$

Similarly for d_{ℓ^∞}:

(i) For all $(a),(b) \in X$, $d_{\ell_\infty}((a),(b)) = 0$ whenever $(a) = (b)$, and $d_{\ell_\infty}((a),(b)) > 0$ whenever $(a) \neq (b)$, since there exists some $i \in \mathbb{N}$ such that $|a_i - b_i| > 0$.

(ii) For all $n \in \mathbb{N}$, we have $|a_n - b_n| = |b_n - a_n|$, so $d_{\ell_\infty}((a),(b)) = d_{\ell_\infty}((b),(a))$.

(iii) For all $(a),(b),(c) \in X$, we have

$$\begin{aligned} d_{\ell\infty}((a),(c)) &= \sup_{n\in\mathbb{N}} |a_n - c_n| \\ &\leq \sup_{n\in\mathbb{N}}(|a_n - b_n| + |b_n - c_n|) \\ &\leq \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n| \\ &\leq d_{\ell_\infty}((a),(b)) + d_{\ell_\infty}((b),(c)), \end{aligned} \tag{1}$$

where we used the triangle inequality of real numbers with respect to $|\cdot|$.

Suppose the sequence (of sequences) $(x^{(m)}) \in X$ converges with respect to the metric d_{ℓ_1}. Then there exists $x \in X$ such that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $m > N$ we have

$$d_{\ell_1}((x^{(m)}),(x)) = \sum_{n=0}^{\infty} |a_n - b_n| < \varepsilon.$$

Then for all $m > N$, we have

$$d_{\ell_\infty}((x^{(m)}),(x)) = \sup_{n\in\mathbb{N}} |a_n - b_n| \leq \sum_{n=0}^{\infty} |a_n - b_n| < \varepsilon,$$

so $(x^{(m)})$ also converges with respect to the metric d_{ℓ_∞}. However, consider the sequence $(x_n^{(m)})_{n=1}^{\infty}$ in $\mathbb{R}$ defined by

$$x_n^{(m)} = \begin{cases} \frac{1}{m}, & 0 \leq n < m, \\ 0, & n \geq m, \end{cases}$$

where $\sum_{m=1}^{\infty} x_n^{(m)} = 1 < \infty$. This sequence converges to the zero sequence (0) in $(\mathbb{R}, d_{\ell^\infty})$ since for all $\varepsilon > 0$ and $m \in \mathbb{N}$, let $N = 1/\varepsilon$, then

$$d_{\ell_\infty}((x^{(m)}),(0)) = \sup_{0\leq n<m} \frac{1}{m} < \frac{1}{N} = \varepsilon$$

whenever $n > N$. However, this sequence does not converge to in $(\mathbb{R}, d_{\ell^1})$ since for all $n \in \mathbb{N}$, pick $i,j \in \mathbb{N}$ such that $i < j$, we have

$$d_{\ell_1}((x^{(i)}),(x^{(j)})) = \sum_{r=0}^{i-1} \frac{1}{i} + \sum_{r=i}^{j-1} \frac{1}{j} = 1 + \frac{j-i-1}{j} > 1.$$

(b) For all distinct $i,j \in \mathbb{N}$, we have $d_{\ell^1}(e^{(i)}, e^{(j)}) = \sum_{k=1}^{\infty} \left| e_k^{(i)} - e_k^{(j)} \right| = 2$. Then for all $x \in X$ and $n \in \mathbb{N}$, we have

$$d_{\ell^1}(x, e^{(n)}) = \sum_{k=1}^{\infty} \left| a_k - e_k^{(n)} \right| \leq \sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} \left| e_k^{(n)} \right| = \sum_{k=1}^{\infty} |a_k| + 1 < \infty.$$

For all $x \in X$, the sequence $\left(e^{(n)}\right)$ is contained in $B(x,\varepsilon)$ with $\varepsilon = d_{\ell^1}\left(e^{(0)}, x\right) + 3$, hence it is <u>bounded</u>. For some $x \in X \backslash \{e^{(n)} \mid n \in \mathbb{N}\}$, hence the complement of $\{e^{(n)} \mid n \in \mathbb{N}\}$ is open, and the set itself is closed. Finally, notice that $d_{\ell^1}\left(e^{(i)}, e^{(j)}\right) = 2$ for any pair of distinct i,j, so the sequence $\left(e^{(n)}\right)$ has no convergent subsequence, hence the $\{e^{(n)} \mid n \in \mathbb{N}\}$ is <u>not compact</u>.

Problem 2 (24 pts). A metric space (X, d) is called totally bounded if for every $\varepsilon > 0$, there exists a natural number n and a finite number of balls

$$B(x^{(1)}, \varepsilon),\ B(x^{(2)}, \varepsilon),\ \ldots,\ B(x^{(n)}, \varepsilon)$$

which cover X (i.e. $X = \bigcup_{i=1}^n B(x^{(i)}, \varepsilon)$).

(a) Show that every totally bounded space is bounded.

(b) Show the following stronger version of Proposition 1.5.5: if (X, d) is compact, then it is complete and totally bounded.

Hint: if X is not totally bounded, then there is some $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Then use Exercise 8.5.20 (on page 182 of Analysis I) to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to construct a sequence which has no convergent subsequence.

(c) Conversely, show that if X is complete and totally bounded, then X is compact.

Hint: if $(x^{(n)})_{n=1}^\infty$ is a sequence in X, use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^\infty$ of $(x^{(n)})_{n=1}^\infty$ for each positive integer j, such that for each j the elements of the sequence $(x^{(n;j)})_{n=1}^\infty$ are contained in a single ball of radius $1/j$. Also ensure that each sequence $(x^{(n;j+1)})_{n=1}^\infty$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^\infty$. Then show that the "diagonal" sequence $(x^{(n;n)})_{n=1}^\infty$ is a Cauchy sequence, and then use the completeness hypothesis.

Solution 2.

(a) Let (X, d) be totally bounded . Then for all $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ and a sequence $\left(x^{(n)}\right)_{n=1}^\infty$ in X such that

$$X = \bigcup_{i=1}^n B(x^{(i)}, \epsilon).$$

Then for all $x, y \in X$, there exists some $i, j \in \{1, 2, \ldots, n\}$ such that $x \in B(x^{(i)}, \varepsilon)$ and $y \in B(x^{(j)}, \varepsilon)$. Let $R = \max\{d(x^{(i)}, x^{(j)}) : i, j \in \{1, 2, \ldots, n\}\}$, by triangle inequality we have

$$d(x, y) \le d(x, x^{(i)}) + d(x^{(i)}, x^{(j)}) + d(x^{(j)}, y) < 2\varepsilon + R.$$

Hence (X, d) is bounded.

(b) Suppose (X, d) is compact. Then by [Tao II] theorem 1.5.7, (X, d) is complete and bounded. Suppose (X, d) is not totally bounded, then there exists $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Fix such ε, choose $x_1 \in X$ and write $B_1 = B(x_1, \varepsilon/2)$, then we may choose $x_2 \in X \backslash B_1$ and write $B_2 = B(x_2, \varepsilon/2)$, and so on. Note that $X \backslash B_i \neq \varnothing$ since they do not cover X. By induction, we obtain a sequence $(x_n) \in X$ such that $d(x_i, x_j) \ge \varepsilon$ for all distinct $i, j \in \mathbb{N}$. Then for all $N \in \mathbb{N}$, there exists some $m, n > N$ such that $d(x_m, x_n) \ge \varepsilon$, so (x_n) has no convergent subsequence, a contradiction. Hence (X, d) is totally bounded.

(c) For each $\varepsilon_n = \frac{1}{n}$, there exists finitely many ε_n-balls that cover X. Collect them into a sequence $(B_{n;k})_{k=1}^{m_n}$. Let $(x^{(n)}) \subseteq X$ be a sequence, then $x^{(1)} \in B_{1;k_1}$ for some $k_1 \in \{1, 2, \ldots, m_1\}$. Since $(x^{(n)})$ is infinite, there exists some subsequence $(x^{(n;1)}) \subseteq (x^{(n)})$ such that $x^{(n;1)} \in B_{1;k_1}$ for all n. Next, since $x^{(n;1)} \in X$, there exists some $k_2 \in \{1, 2, \ldots, m_2\}$ such that $x^{(n;1)} \in B_{2;k_2}$. Again, since $(x^{(n;1)})$ is infinite, there exists some subsequence $(x^{(n;2)}) \subseteq (x^{(n;1)})$ such that $x^{(n;2)} \in B_{2;k_2}$ for all n. Continuing this process, we obtain a sequence of subsequences $(x^{(n;j)})_{n=1}^\infty$ for each $j \in \mathbb{N}$ such that for all n, $x^{(n;j)} \in B_{j;k_j}$ for some $k_j \in \{1, 2, \ldots, m_j\}$, and $(x^{(n;j+1)}) \subseteq (x^{(n;j)})$. Consider the diagonal sequence $(x^{(n;n)})$, then for all $\varepsilon > 0$, let $N = \text{floor}\left(\frac{1}{\varepsilon}\right)$, and for all $k, l > N$, we have $d\left(x^{(k;k)}, x^{(l;l)}\right) \le \frac{1}{N} < \varepsilon$. Hence $(x^{(n;n)})$ is Cauchy, and by completeness of (X, d), there exists some $x \in X$ such that $x^{(n;n)} \to x$. Thus (X, d) is compact.

Remark. The sub-problems (b) and (c) becomes easy if we use the fact from Problem 3: a metric space is compact if and only if every open cover has a finite subcover.

Problem 3 (16 pts)**.**

(a) A metric space (X, d) is compact if and only if every sequence in X has at least one limit point in X.

(b) Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact. Hint: If X is not compact, then by part (a) there is a sequence $(x^{(n)})_{n=1}^{\infty}$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing x which contains at most finitely many elements of this sequence. Now use the hypothesis.

Solution 3.

(a) Suppose every sequence in X has at least one limit point in X. Let $(x^{(n)}) \in X$ be a sequence, then for all $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists some limit point$x \in X$ and $n > N$ such that $d(x^{(n)}, x) < \varepsilon$. For all N, collect the corresponding n_N, so $(x^{(n)})$ has a convergent subsequence $(x^{(n_N)}) \to x \in X$. Conversely, suppose (X, d) is compact. Then for all sequence $(x^{(n)}) \in X$, there exists some subsequence $(x^{(n_k)})$ which converges to some $x \in X$. Then for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $d(x^{(n_k)}, x) < \varepsilon$ whenever $k > M$. Hence x is a limit point of $(x^{(n)})$.

(b) Suppose that X is not compact. Then by part (a), there exists some sequence $(x^{(n)})_{n=1}^{\infty} \in X$ with no limit points in X. For all $x \in X$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon)$ contains finitely many elements of this sequence. Consider the collection of open sets

$$R = \{B(x, \varepsilon) : x \in X\}, \quad X = \bigcup_{U \in R} U.$$

R is an open cover of X, hence there exists some finite subcover $R_0 \subseteq R$ such that

$$X = \bigcup_{U \in R_0} U.$$

However, since each $U \in R_0$ contains finitely many elements of $(x^{(n)})$, we conclude there are only finitely many elements of $(x^{(n)})$ in X, a contradiction. Thus X is compact.

Remark. We have shown one direction of the result, where the other direcrtion is [Tao II] Theorem 1.5.8: *a metric space is compact if and only if every open cover has a finite subcover.*

Problem 4 (10 pts)**.** Let (X, d) be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in X with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus

$$\bigcap_{\alpha \in F} K_\alpha \neq \varnothing \quad \text{for all finite } F \subseteq I.$$

(This property is known as the finite intersection property.)

Show that the entire collection has non-empty intersection, thus

$$\bigcap_{\alpha \in I} K_\alpha \neq \varnothing.$$

Show by counterexample that this statement fails if X is not compact.

Solution 4. Since (X, d) is compact, every open cover of X has a finite subcover. Suppose R is a collection of open sets that cover X, then

$$X = \bigcup_{U \in R_0} U,$$

for some $R_0 \subseteq R$ which is a finite subcollection. Take the complement of the above equation, we have

$$\varnothing = X^c = \left(\bigcup_{U \in R_0} U \right)^c = \bigcap_{U \in R_0} U^c.$$

Since U is open, U^c is closed. Thus the claim is true.

The statement fails if X is not compact. Consider the metric space $(\mathbb{N}, d)$ where $d(x, y) = |x - y|$. Since $\mathbb{N}$ is not complete, it is not compact. Let $K_n = [n, \infty)$, then for all finite $F \subseteq \mathbb{N}$, we have

$$\bigcap_{n \in F} K_n = \bigcap_{n \in F} [n, \infty) = [\max F, \infty) \neq \varnothing.$$

However, the entire collection has empty intersection:

$$\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} [n, \infty) = \varnothing.$$

Problem 5. (a) Let (X, d) be a metric space, and let $(E, d|_{E \times E})$ be a subspace of (X, d). Let $\iota_{E \to X} : E \to X$ be the inclusion map, defined by setting

$$\iota_{E \to X}(x) := x \quad \text{for all } x \in E.$$

Show that $\iota_{E \to X}$ is continuous.

(b) Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E \times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus

$$f|_E(x) := f(x) \quad \text{when } x \in E.$$

If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.)

Conclude that if f is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity.

Hint: use part (a).

(c) Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Suppose that the image $f(X)$ of X is contained in some subset $E \subseteq Y$ of Y. Let $g : X \to E$ be the function which is the same as f but with the codomain restricted from Y to E, thus $g(x) = f(x)$ for all $x \in X$.

Note on codomain: The codomain of a function is the declared target set of the function, in contrast to the image (or range), which is the set of values the function actually takes. So while f is originally defined with codomain Y, its values all lie in the smaller set $E \subseteq Y$. Therefore, one can equivalently regard f as a function $g : X \to E$. The metric on E is the one induced from Y, i.e. $d_Y|_{E \times E}$.

Show that for any $x_0 \in X$, f is continuous at x_0 if and only if g is continuous at x_0. Conclude that f is continuous if and only if g is continuous.

(Thus the notion of continuity is not affected if one restricts the codomain of the function.)

Solution 5.

(a) The preimage of an open ball in X is $\iota^{-1}\left(B_{(X,d)}(x,\varepsilon)\right) = B_{(E,d|_{E\times E})}(x,\varepsilon)$, hence it is open. By [Tao II] theorem 2.1.5, ι is continuous.

(b) Suppose f is continuous at $x_0 \in E$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. Since $\iota_{E\to X}$ is continuous, we have $d_Y(f|_E(x), f|_E(x_0)) = d_Y(f(\iota(x)), f(\iota(x_0))) < \varepsilon$ whenever $d_X(x, x_0) < \delta$, hence $f|_E$ is continuous at x_0. Conversely, suppose $f|_E$ is continuous at $x_0 \in E$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, $d_X(x, x_0) < \delta \implies d_Y(f|_E(x), f|_E(x_0)) < \varepsilon$. Since $\iota_{E\to X}$ is continuous, we have $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$, hence f is continuous at x_0. The converse is not true, however, since the floor function $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{R}$ is continuous on $E = [0, 1)$ but not on $\mathbb{R}$.

(c) Suppose f is continuous at $x_0 \in X$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. Then $d_Y|_{E\times E}(g(x), g(x_0)) = d_Y(f(x), f(x_0)) < \varepsilon$, hence g is continuous at x_0. Conversely, suppose g is continuous at $x_0 \in E$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, $d_Y(g(x), g(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. Since $g(x) = f(x)$ for all $x \in E$, we have $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$, hence f is continuous at x_0.

Problem 6. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \mapsto Y$ is a function from X to Y.

(a) Prove that f is continuous on X if, and only if,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset A of X.

(b) Prove that f is continuous on X if and only if f is continuous on every compact subset of X.

Hint: If $x_n \to p$ in X, the set $\{p, x_1, x_2, \dots\}$ is compact.

Solution 6.

(a) Suppose f is continuous on X. Let $A \subseteq X$, for all $x \in \overline{A}$, there exists some sequence $(x^{(n)}) \in A$ such that $x^{(n)} \to x$ under d_X. Then by continuity of f and [Tao II] theorem 2.1.4, we have $f(x^{(n)}) \to f(x)$ under d_Y. Since $f(x^{(n)}) \in f(A)$, we have $f(x^{(n)}) \to f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$. Conversely, suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $F \subseteq Y$ be a closed subset, then $F = \overline{F}$ and $f(f^{-1}(F)) \subseteq F$. Then by assumption $f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$, and $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. Hence $f^{-1}(F)$ is closed, and f is continuous by [Tao II] theorem 2.1.5.

(b) Suppose f is continuous on X, then for all compact subset $K \subseteq X$, the restriction of f to K is continuous on K. Conversely, suppose f is continuous on every compact subset of X. Let $F \subseteq Y$ be closed and let $x \in \overline{f^{-1}(F)}$, then there exists a sequence $(x^{(n)}) \in X$ such that $x^{(n)} \to x \in X$. The set $K = \{x, x^{(1)}, x^{(2)}, \dots\}$ is compact since every sequence in K has a convergent subsequence. By assumption, f is continuous on K, so by [Tao II] theorem 2.1.4, we have $f(x^{(n)}) \to f(x) \in F$, hence $x \in f^{-1}(F)$. Thus $f^{-1}(F)$ is closed, and by the same theorem f is continuous on X.