Math 2213 Introduction to Analysis I

Homework 7 Due November 7 (Friday), 2025

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Exercise 1 (15 pts). Assume that (S, d) is a metric space, and let $f_n, f : S \to \mathbb{R}$ be real-valued functions. Suppose that $f_n \to f$ uniformly on S, and there exists a constant $M > 0$ such that

$$|f_n(x)| \le M \quad \text{for all } x \in S \text{ and all } n.$$

Let $g : \overline{B(0; M)} \to \mathbb{R}$ be continuous, where

$$B(0; M) = \{y \in \mathbb{R} : |y| < M\}.$$

Define

$$h_n(x) = g(f_n(x)), \qquad h(x) = g(f(x)), \quad x \in S.$$

Prove that $h_n \to h$ uniformly on S.

Solution 1. Since g is continuous on the closed interval $\overline{B(0; M)}$, by previous homework it is uniformly continuous on this interval. Therefore, for the given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|y_1 - y_2| < \delta$ for any $y_1, y_2 \in \overline{B(0; M)}$, we have $|g(y_1) - g(y_2)| < \varepsilon$. Since $f_n \rightrightarrows f$ uniformly on S, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ and all $x \in S$, we have $|f_n(x) - f(x)| < \delta$. Hence, for all $n \ge N$ and all $x \in S$, we have

$$|h_n(x) - h(x)| = |g(f_n(x)) - g(f(x))| < \varepsilon,$$

since $|f_n(x) - f(x)| < \delta$. Therefore, $h_n \rightrightarrows h$ on S.

Exercise 2 (15 pts). Let $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$. Let g be continuous on $[0, 1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

Solution 2. The sequence $\{f_n\}$ converges to the function

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

We claim that $g(x)x^n \rightrightarrows 0$, and by continuity of g at 1, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(x) - g(1)| < \varepsilon$ whenever $|x - 1| < \delta$. We have

$$|g(x)x^n - 0| = |g(x)x^n| \le |g(x)| = |g(x) - g(1)| < \varepsilon$$

whenever $|x - 1| \le \delta$. Next, consider the case when $x \in [0, 1 - \delta]$. Since g is continuous on $[0, 1]$, it is bounded by $M = \max\{g\} > 0$ on $[0, 1]$. Thus, for $x \in [0, 1 - \delta]$, we have

$$|g(x)x^n - 0| = |g(x)x^n| \le M(1 - \delta)^n.$$

Since $0 < 1 - \delta < 1$, we can choose $N \in \mathbb{N}$ such that $M(1 - \delta)^N < \varepsilon$. Therefore, for all $n \ge N$ and all $x \in [0, 1 - \delta]$, we have $|g(x)x^n - 0| < \varepsilon$. Combining both cases proves uniform convergence of the sequence $\{g(x)x^n\}$.

Exercise 3 (15 pts). Assume that $g_{n+1}(x) \le g_n(x)$ for each x in T and each $n = 1, 2, \ldots$, and suppose that $g_n \to 0$ uniformly on T. Prove that

$$\sum (-1)^{n+1} g_n(x)$$

converges uniformly on T.

Solution 3. Since $g_n \rightrightarrows 0$ and $g_{n+1}(x) \le g_n(x)$, for all n, we have $g_n(x) \ge 0$ for all $x \in T$. Fix $x \in T$, the n-th partial sum $S_n = \sum_{k=1}^{n} g_k(x)$ satisfies the following inequalities:

$$S_{2m+1}(x) \le S_{2m+3}(x) \le S_{2m+2}(x), \quad S_{2m}(x) \le S_{2m+2}(x) \le S_{2m+1}(x).$$

Hence, every later partial sum $S_{m \ge n+1}$ lies in the interval $[S_{n+1}(x), S_n(x)]$ or $[S_n(x), S_{n+1}(x)]$. Therefore, for all $m > n$, we have

$$|S_m(x) - S_n(x)| \le |S_{n+1}(x) - S_n(x)| = g_{n+1}(x).$$

Since $g_n \rightrightarrows 0$ on T, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ and all $x \in T$, we have $g_{n+1}(x) < \varepsilon$. Therefore, for all $m > n \ge N$ and all $x \in T$, we have $|S_m(x) - S_n(x)| < \varepsilon$. Hence (S_n) is Cauchy on T, and the pointwise limit $S(x) = \lim_{n\to\infty} S_n(x)$ exists for each $x \in T$. Then

$$|S(x) - S_n(x)| = \lim_{m\to\infty} |S_m(x) - S_n(x)| \le g_{n+1}(x) < \varepsilon, \quad \text{as } m \to \infty,$$

for all $n \ge N$ and all $x \in T$. Therefore, $S_n \rightrightarrows S$ on T.

Exercise 4 (15 pts). Let

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R},\ n = 1, 2, \ldots$$

Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x but that $f'(0) \ne g(0)$. For what values of x is $f'(x) = g(x)$?

(b) In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly?

(c) In what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

Solution 4. Since $f_n(0) = 0$ for all n, suppose $x \ne 0$, so

$$0 \le |f_n(x)| = \left| \frac{x}{1 + nx^2} \right| \le \left| \frac{1}{nx} \right| \to 0 \quad \text{as } n \to \infty.$$

By the Squeeze Theorem, the sequence $\{f_n\}$ converges to $f = 0$. On the other hand, we have

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

Since $f_n'(0) = 1$, suppose $x \ne 0$, then

$$|f_n'(x)| = \left| \frac{1 - nx^2}{(1 + nx^2)^2} \right| \le \left| \frac{nx^2}{n^2x^4} \right| = \left| \frac{1}{nx^2} \right| \to 0 \quad \text{as } n \to \infty.$$

By the Squeeze Theorem, the sequence $\{f_n'\}$ converges to $g(x) = 0$ for $x \ne 0$ and $g(0) = 1$.

(a) By the above calculation, since $1 + nx^2 > 0$ for all $x \in \mathbb{R}$, $f'(x)$ exists for every x. However, $f'(0) = 0 \ne g(0) = 1$. For $x \ne 0$, we have $f'(x) = g(x) = 0$.

(b) We have

$$|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2}\right| \le \sup_{x\in\mathbb{R}}\left|\frac{x}{1+nx^2}\right| \le \left.\frac{|x|}{1+nx^2}\right|_{x=n^{-1/2}} = \frac{1}{2\sqrt{n}},$$

Given $\varepsilon > 0$, choose $N = \frac{1}{4\varepsilon^2}$, then for all $n > N$ we have $\frac{1}{2\sqrt{n}} < \varepsilon$. Therefore, $f_n \rightrightarrows f$ on $\mathbb{R}$.

(c) For any interval $[a, b]$ not containing the origin, where without loss of generality we set $0 < a < b$. For all $\varepsilon > 0$, let $N = \frac{1}{\varepsilon a^2}$, then for all $n \ge N$ and all $x \in [a, b]$, we have

$$|f_n'(x) - g(x)| = \left|\frac{1-nx^2}{(1+nx^2)^2}\right| \le \frac{1}{nx^2} \le \frac{1}{na^2} < \varepsilon.$$

Therefore, $f_n' \rightrightarrows g$ on any interval not containing 0. Next, consider an open interval with 0 as an end point. Without loss of generality, let it be $(0, b)$, $b > 0$. Since $\lim_{x\to 0^+} f_n'(x) = 1$ for all n, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in (0, \delta)$, we have

$$\left|\frac{1-nx}{(1+nx^2)^2}\right| > 1 - \varepsilon \implies \sup_{x\in(0,b)} |f_n'(x) - g(x)| = 1$$

for all n. Hence, convergence is not uniform on $(0, b)$. Therefore, $f_n' \rightrightarrows g$ exactly on the intervals $I \subseteq \mathbb{R}$ where $\inf_{x\in I} |x| > 0$.

Exercise 5 (15 pts)**.** Prove that

$$\sum x^n(1-x)$$

converges pointwise but not uniformly on $[0, 1]$, whereas

$$\sum(-1)^n x^n(1-x)$$

converges uniformly on $[0, 1]$. This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.

Solution 5.

1. $\sum x^n(1-x)$: If $x = 0$ or $x = 1$, then $\sum x^n(1-x) = 0$ for all n. Suppose $x \in (0, 1)$, let

$$f_n = \sum_{k=1}^{n} x^k(1-x) = \frac{x(1-x)(1-x^n)}{1-x} = x(1-x^n) \to x \quad \text{as } n \to \infty.$$

be the n-th partial sum. Then, for some $x \in \mathbb{R}$ and $\varepsilon > 0$, let $N_x = \frac{\log \varepsilon}{\log x} - 1$, we have

$$|f_n(x) - x| = |x^{n+1}| < \varepsilon, \quad \text{whenever } n > N_x.$$

Since N_x is unbounded for $x \in \mathbb{R}$, the convergence is not uniform on $[0, 1]$.

2. $\sum(-1)^n x^n(1-x)$: If $x = 0$ or $x = 1$, then $\sum x^n(1-x) = 0$ for all n. Suppose $x \in (0, 1)$, let

$$g_n(x) = \sum_{k=1}^{n}(-1)^k x^k(1-x) = x(1-x)\frac{1-(-x)^{n+1}}{1+x} \to \frac{x(1-x)}{1+x} \quad \text{as } n \to \infty.$$

Then, for all $\varepsilon > 0$, let $N = 1/(e\varepsilon) - 2$, then whenever $n > N$, we have

$$\left|g_n(x) - \frac{x(1-x)}{1+x}\right| = \left|\frac{x(1-x)}{1+x}x^{n+1}\right| < (1-x)x^{n+1} < \frac{1}{(n+2)\left(1+\frac{1}{n+1}\right)^{n+1}} < \varepsilon.$$

Here we used

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)x^{n+1}\right) = x^n\left(n+1-(n+2)x\right) = 0 \implies x = \frac{n+1}{n+2}.$$

Exercise 6 (15 pts). Let

$$f_n(x) = \frac{1}{n}e^{-n^2x^2}, \quad x \in \mathbb{R},\ n = 1, 2, \ldots$$

Prove that $f_n \to 0$ uniformly on $\mathbb{R}$, that $f_n' \to 0$ pointwise on $\mathbb{R}$, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

Solution 6.

1. $f_n \rightrightarrows 0$ on $\mathbb{R}$: For any $x \in \mathbb{R}$ and $\varepsilon > 0$, let $N = \varepsilon^{-1}$, then for all $n > N$ we have
$$|f_n(x) - 0| = \left|\frac{1}{n}e^{-n^2x^2}\right| \le \frac{1}{n} < \varepsilon.$$

2. $f_n' \to 0$ on $\mathbb{R}$: For any $x \in \mathbb{R}$, we have $f_n'(x) = -2xe^{-n^2x^2}$. When $x = 0$, $f_n = 0$. So consider $x \in \mathbb{R} \setminus \{0\}$, let $N_x = \frac{1}{x}\sqrt{\log(2|x|/\varepsilon)}$ if $|x| > \frac{\varepsilon}{2}$ and $N_x = \frac{1}{|x|}$ otherwise. Then for all $n > N_x$ we have
$$|f_n'(x) - 0| = \left|2xe^{-n^2x^2}\right| < \varepsilon.$$
Hence $f_n' \to 0$ pointwise. However, if zero is contained in the interval, $\lim_{x\to 0} N_x = \lim_{x\to 0}\frac{1}{x^2}$ does not exist, so convergence is not uniform.

Exercise 7 (10 pts). Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0, 1]$ and assume that $f_n \to f$ uniformly on $[0, 1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Solution 7. For each n, notice that

$$\int_0^{1-1/n} f_n = \int_0^{1-1/n} f + \int_0^{1-1/n} (f_n - f).$$

Hence,

$$\left|\int_0^{1-1/n} f_n - \int_0^1 f\right| \le \left|\int_0^{1-1/n} (f_n - f)\right| + \left|\int_{1-1/n}^1 f\right|.$$

Since $f_n \rightrightarrows f$ on $[0, 1]$, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$ and all $x \in [0, 1]$, we have $|f_n(x) - f(x)| < \varepsilon/2$. Therefore, for all $n \ge N_1$, we have

$$\left|\int_0^{1-1/n} (f_n - f)\right| \le \int_0^{1-1/n} |f_n - f| < \frac{\varepsilon}{2}.$$

On the other hand, since f is continuous on $[0, 1]$, it is integrable on $[0, 1]$. Thus, there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, we have

$$\left|\int_{1-1/n}^1 f\right| < \frac{\varepsilon}{2}.$$

Therefore, for all $n \ge \max\{N_1, N_2\}$, we have

$$\left|\int_0^{1-1/n} f_n - \int_0^1 f\right| < \varepsilon \implies \lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f.$$

You can do the following problems to practice.
You don't have to submit the following problems.

Exercise 8 (Optional). Prove that the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges uniformly on every half-infinite interval

$$1 + h \leq s < +\infty,$$

where $h > 0$. Show that the equation

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

is valid for each $s > 1$, and obtain a similar formula for the kth derivative $\zeta^{(k)}(s)$.

Solution 8.

Exercise 9 (Optional). If r is the radius of convergence of

$$\sum a_n (x - x_0)^n,$$

where each $a_n \neq 0$, show that

$$\liminf_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \leq r \leq \limsup_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Solution 9.

Exercise 10 (Optional). Prove that the series

$$\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right)$$

converges pointwise but not uniformly on $[0, 1]$.

Solution 10.

Exercise 11 (Optional). Prove that

$$\sum_{n=1}^{\infty} a_n \sin nx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \cos nx$$

are uniformly convergent on $\mathbb{R}$ if

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

Solution 11.

Exercise 12 (Optional)**.** Let $\{a_n\}$ be a decreasing sequence of positive terms. Prove that the series

$$\sum a_n \sin nx$$

converges uniformly on $\mathbb{R}$ if and only if $na_n \to 0$ as $n \to \infty$.

Solution 12.