Math 2213 Introduction to Analysis I

Homework 8 Due November 14 (Friday), 2025

物理、數學三 黃紹凱 B12202004

November 16, 2025

Exercise 1 (25 points). Give examples of a formal power series

$$\sum_{n=0}^{\infty} c_n x^n$$

centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$;

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

Solution 1.

(a) The series

$$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

has radius of convergence 1 and diverges at both $x = 1$ and $x = -1$.

(b) The series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

has radius of convergence 1, diverges at $x = 1$ but converges at $x = -1$.

(c) The series

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

has radius of convergence 1, diverges at $x = 1$ but converges at $x = -1$.

(d) The series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$$

has radius of convergence 1 and converges at both $x = 1$ and $x = -1$.

(e) The series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

converges pointwise on $(-1, 1)$ as shown above. However, since it is unbounded, it does not converge uniformly on $(-1, 1)$.

Exercise 2 (**Tao II Ex. 4.2.7.**, 25 points)**.** Let $m \geq 0$ be a positive integer, and let $0 < r$ be real numbers. Prove the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$. Using Proposition 4.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and all $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

Solution 2. First we show that the identity is true:

$$\frac{r}{r-x} = \frac{1}{1-\frac{x}{r}} = \sum_{n=0}^{\infty} \left(\frac{x}{r}\right)^n = \sum_{n=0}^{\infty} x^n r^{-n},$$

for $-1 < \frac{x}{r} < 1$, which is equivalent to $x \in (-r, r)$. By Proposition 4.2.6, there exists $r > 0$ such that the power series on the right is m times differentiable on $(-r, r)$. Differentiate both sides m times, we have

$$\frac{d^m}{dx^m}\left(\frac{r}{r-x}\right) = m!\,\frac{r}{(r-x)^{m+1}},$$

$$\frac{d^m}{dx^m}\left(\sum_{n=0}^{\infty} x^n r^{-n}\right) = \sum_{n=0}^{\infty} r^{-(n+m)} \frac{(n+m)!}{n!} x^n = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}.$$

Equating both sides gives the desired identity for all $m \geq 0$ and $x \in (-r, r)$.

Exercise 3 (25 points)**.** Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real analytic at a and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r,\ a+r)$. Let $(b-s,\ b+s)$ be any subinterval of $(a-r,\ a+r)$ for some $s > 0$.

(a) Prove that $|a-b| \leq r-s$, so in particular $|a-b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \leq C(r-\varepsilon)^{-n}$ for all integers $n \geq 0$. (Hint: what do we know about the radius of convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$?)

(c) Show that the numbers $d_0, d_1, \ldots$, given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} (b-a)^{n-m} c_n \qquad \text{for all integers } m \geq 0,$$

are well-defined, in the sense that the above series is absolutely convergent. (Hint: use (b) and the comparison test, Corollary 7.3.2, followed by Exercise 4.2.7.)

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \leq C(s-\varepsilon)^{-m}$$

for all integers $m \geq 0$. (Hint: use the comparison test, and Exercise 4.2.7.)

(e) Show that the power series $\sum_{m=0}^{\infty} d_m(x-b)^m$ is absolutely convergent for $x \in (b-s,\, b+s)$ and converges to $f(x)$. (You may need Fubini's theorem for infinite series, Theorem 8.2.2 of Analysis I, as well as Exercise 4.2.5. One may also need to use a variant of the d_m in which the c_n are replaced by $|c_n|$.) Note. You can use Exercise 4.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

for any real number x.

(f) Conclude that f is real analytic at b, and thus analytic at every point in $(a-r,\, a+r)$.

Solution 3.

Exercise 4 (25 points)**.**

(a) If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1^-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$.)

(b) If each $a_n \geq 0$ and if $\lim_{x\to 1^-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has sum A.

Solution 4.

You can do the following problems to practice.
You don't have to submit the following problems.

Exercise 5 (Optional). Let the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converge for $-1 < x < 1$. For each n, define the partial sum

$$s_n = \sum_{k=0}^{n} a_k, \qquad \sigma_n = \sum_{k=0}^{n} k|a_k|.$$

Suppose that $\lim_{x\to 1^-} f(x) = S$ and $\lim_{n\to\infty} na_n = 0$. In this problem, you will show that the series $\sum_{n=0}^{\infty} a_n$ converges and that its sum is S.

(a) **Preliminary Identity.** Show that for any $x \in (0, 1)$,

$$s_n - f(x) = \sum_{k=0}^{n} a_k(1 - x^k) - \sum_{k=n+1}^{\infty} a_k x^k.$$

(b) **Bounding the First Sum.** Show that for all $m \geq 1$ and $x \in (0, 1)$,

$$1 + x + \cdots + x^{m-1} \leq \frac{1}{1-x},$$

and deduce that

$$|1 - x^k| = (1 - x)(1 + x + \cdots + x^{k-1}) \leq k(1 - x).$$

(c) **Application of the Bound.** Use part (b) to prove that for $x \in (0, 1)$,

$$\left| \sum_{k=0}^{n} a_k(1 - x^k) \right| \leq (1 - x)\sigma_n.$$

(d) **Estimate of the Tail.** Use the assumption that $\lim_{n\to\infty} n|a_n| = 0$ to show that for any $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$n|a_n| < \frac{\varepsilon}{3}.$$

Then prove that for such n and all $x \in (0, 1)$,

$$\left| \sum_{k=n+1}^{\infty} a_k x^k \right| \leq \frac{\varepsilon}{3(1-x)}.$$

(e) **Putting the Estimates Together.** Combine parts (a)–(d) to show that for all $n \geq N$ and $x \in (0, 1)$,

$$|s_n - S| \leq |f(x) - S| + (1 - x)\sigma_n + \frac{\varepsilon}{3(1-x)}.$$

(f) **Strategic Choice of x.** Let $x = x_n = 1 - \frac{1}{n}$. Use part (e) to show that when n is sufficiently large,

$$|s_n - S| < \varepsilon.$$

Conclude that $s_n \to S$, and therefore

$$\sum_{n=0}^{\infty} a_n = S.$$

Solution 5.

Exercise 6 (Optional)**.**

(1) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Show that the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is $+\infty$.

(2) Suppose that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ has radius of convergence $R < +\infty$. What can we say about the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$?

Solution 6.

Exercise 7 (Optional)**.** Let $(a_n)_{n\geq 1}$ be a sequence of nonzero real numbers such that

$$\frac{|a_{n+2}|}{|a_n|} \xrightarrow[n\to\infty]{} 2.$$

Show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{\sqrt{2}}$.

Solution 7.