Math 2213 Introduction to Analysis I

Homework 9 Due November 21 (Friday), 2025

物理三 黃紹凱 B12202004

November 22, 2025

Exercise 1 (15 points). Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $S_n = \sum_{k=0}^{n} a_k$ be the partial sums of $\sum a_n$. Denote the radius of convergence of $\sum_{n=0}^{\infty} S_n x^n$ by r.

(a) Show that $r \leq R$.

(b) Show that $\min\{1, R\} \leq r$. Hint: The power series $\sum_{n=0}^{\infty} S_n x^n$ can be seen as the Cauchy product between $\sum_{n=0}^{\infty} a_n x^n$ and a specific power series that you need to choose.

Solution 1.

(a)

(b) Let $(b_n) = (1, 1, \dots)$ be a sequence of all ones. Then

$$\sum_{n=0}^{\infty} S_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n = \left(\sum_{n=0}^{\infty} a_n x^n \right) * \left(\sum_{n=0}^{\infty} b_n x^n \right),$$

where $*$ denotes the Cauchy product. Since the radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ is 1, we have $r \geq \min\{1, R\}$.

Exercise 2 (30 points). For each real t, define

$$f_t(x) = \begin{cases} \dfrac{x e^{xt}}{e^x - 1}, & x \in \mathbb{R},\ x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) Show that there exists $\delta > 0$ such that f_t admits a power series expansion in x for all $|x| < \delta$.

Hint. Write

$$f_t(x) = e^{xt} g(x),$$

where

$$g(x) = \begin{cases} \dfrac{x}{e^x - 1}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Both e^{xt} and $g(x)$ are analytic near 0. Also $g(x) = \frac{1}{h(x)}$ where $h(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$ and we can express it as an power series in x. Then may use the fact that if h is analytic on $\mathbb{R}$ and $h(0) \neq 0$, then $1/h$ is analytic on a smaller interval $(-\delta, \delta)$.

(b) Define $P_0(t), P_1(t), P_2(t), \dots$ by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \qquad x \in (-\delta, \delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

to prove that

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{\,n-k}.$$

(Hint: $f_t(x) = e^{tx} f_0(x)$ and $f_0(x) = g(x)$.) This shows that each function P_n is a polynomial. These are the Bernoulli polynomials. The numbers $B_n := P_n(0)$ $(n = 0, 1, 2, \ldots)$ are called the Bernoulli numbers. Derive the following further properties:

(c) $B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0,$ if $n = 2, 3, \ldots$

(d) $P_n'(t) = n\, P_{n-1}(t), \quad$ if $n = 1, 2, \ldots$

(e) $P_n(t+1) - P_n(t) = n\, t^{n-1}, \quad$ if $n = 1, 2, \ldots$

(f) $P_n(1-t) = (-1)^n P_n(t)$

(g) $B_{2n+1} = 0, \quad$ if $n = 1, 2, \ldots$

(h)

$$1^n + 2^n + \cdots + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}, \qquad (n = 2, 3, \ldots).$$

Solution 2.

(a) Since both e^{xt} and $g(t)$ are analytic near 0, we have $h(x) = \frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, which is convergent for all $x \in \mathbb{R}$. Note that $h(0) = 1 \neq 0$, thus there exists some $\delta > 0$ such that $g(x) = \frac{1}{h(x)}$ is analytic on $(-\delta, \delta)$. Therefore, $f_t(x) = e^{xt} g(x)$ is analytic on $(-\delta, \delta)$.

(b) Using $f_t(x) = e^{tx} f_0(x)$, by the Cauchy product formula, we have

$$\begin{aligned} f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} &= e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} \\ &= \left(\sum_{m=0}^{\infty} \frac{(tx)^m}{m!} \right) \left(\sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} P_k(0) \frac{x^k}{k!} \frac{(tx)^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{x^n}{n!} \right) P_k(0) t^{n-k}. \end{aligned}$$

Comparing the coefficients of x^n in the sense of a formal power series, we have

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k}.$$

(c) The Bernoulli numbers are given by

$$g(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Compare this with the Taylow expansion, we have $\lim_{x\to 0} g^{(n)}(x) = B_n$. The first few derivatives and their limits are

$$\begin{aligned} g(x) &= \frac{x}{e^x - 1}, \quad \lim_{x\to 0} g(x) = 1, \\ g'(x) &= \frac{e^x(x-1) + 1}{(e^x - 1)^2}, \quad \lim_{x\to 0} g'(x) = -\frac{1}{2}, \end{aligned}$$

and so on. Hence, $B_0 = 1$ and $B_1 = -\frac{1}{2}$. Next, we will work in the ring of formal power series $\mathbb{R}[[x]]$. We have

$$e^x - 1 = \sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)!},$$

thus, by the Cauchy product of power series,

$$\begin{aligned} x &= \left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)!} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} B_j \frac{x^j}{j!} \frac{x^{k-j+1}}{(k-j+1)!} \right) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} B_j \frac{(k+1)!}{j!(k-j+1)!} \frac{x^{k+1}}{(k+1)!} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k+1}{j} B_j \right) \frac{x^{k+1}}{(k+1)!} \end{aligned}$$

Reindex $k = n-1$ and $j = k$, then

$$x = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{x^n}{n!} \implies \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n = 2, 3, \ldots.$$

(d) Differentiating both sides of (b) in $\mathbb{R}[[t]]$, we have

$$\begin{aligned} P_n'(t) &= \sum_{k=0}^{n} \binom{n}{k} P_k(0)(n-k)t^{n-k-1} = \sum_{k=0}^{n} \frac{n!}{k!(n-k-1)!} t^{n-k-1} \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} P_k(0) t^{n-1-k} = n P_{n-1}(t). \end{aligned}$$

(e) By the formula in (b), we have

$$\begin{aligned} P_n(t+1) - P_n(t) &= \sum_{k=0}^{n} \binom{n}{k} P_k(0)(t+1)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} P_k(0) \left((t+1)^{n-k} - t^{n-k} \right) \end{aligned}$$

(f) Substitute $1-t$ into the generating function of Bernoulli polynomials, we have

$$\sum_{n=0}^{\infty} P_n(1-t) \frac{x^n}{n!} = \frac{xe^{(1-t)x}}{e^x - 1} = \frac{xe^{-tx}}{e^{-x} - 1} = \frac{(-x)e^{t(-x)}}{1 - e^{-x}} = \sum_{n=0}^{\infty} (-1)^n P_n(t) \frac{x^n}{n!}.$$

(g) Consider the function $\tilde{g}(x) = g(x) - P_1(0)x = g(x) - B_1 x$. We have

$$\tilde{g}(x) = \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x(e^{x/2} + e^{-x/2})}{2(e^{x/2} - e^{-x/2})} = \frac{x}{2} \coth\left(\frac{x}{2}\right)$$

is even, thus all odd derivatives of $\tilde{g}$ at 0 are zero. Therefore, for $n \geq 1$, we have

$$B_{2n+1} = g^{(2n+1)}(0) = \tilde{g}^{(2n+1)}(0) = 0.$$

(h) The first and third equalities follow from (e), and the second is due to the telescoping sum:

$$\sum_{j=1}^{k-1} j^n = \sum_{j=1}^{k-1} \frac{P_n(j+1) - P_n(j)}{n} = \frac{P_n(k) - P_n(1)}{n} = \frac{P_n(k) - P_n(0)}{n},$$

Exercise 3 (Tao II Exercise 4.2.7., 15 points)**.** Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

(Hint: first show that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3, \ldots$.)

Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational. *(Hint: prove by contradiction.)*

Solution 3. First, we show that $(n+k)! > 2^k n!$ for all $k \in \mathbb{N}$ and $n \geq 3$ by induction. For $k = 1$, we have $(n+1)! = (n+1)n! > 2n!$. Assume it holds for k, then for $k+1$, we have $(n+k+1)! = (n+k+1)\cdots(n+1)n! > 2^{k+1}n!$. Thus, the inequality holds for all $k \in \mathbb{N}$. Then,

$$0 < \sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \sum_{k=1}^{\infty} \frac{1}{2^k n!} = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{n!}.$$

Suppose there exists some $n \geq 3$ such that $n!e$ is an integer. Then,

$$n!e = n! \sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{n} (n-k)! + n! \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

is an integer, and hence

$$0 < n! \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{n!}{n!} = 1$$

is an integer, a contradiction. Therefore, $n!e$ is not an integer for any $n \geq 3$. If e were rational, then $q!e$ is an integer, where $q \in \mathbb{N}$ is the denominator of e, contradicting the previous result. Thus, e is irrational.

Exercise 4 (Tao II Exercise 4.5.6, 10 points)**.** Prove that the natural logarithm function $\ln x$ is real analytic on $(0, +\infty)$. Hint: For any $a > 0$, consider the change of variable $y = x - a$.

Solution 4. To show ln is real analytic on $(0, \infty)$, it suffices to show that for every $a > 0$, there is a power series centered at a that equals $\ln x$ on some interval around a. From Tao II Theorem 4.5.6 (e), we have $\ln(1+x)$ is real analytic at $x = 0$, such that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad x \in (-1, 1),$$

with radius of convergence 1. For any $a > 0$, let $y = x - a$, then

$$\ln x = \ln(a+y) = \ln a + \ln\left(1 + \frac{y}{a}\right) = \ln a + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{y}{a}\right)^n, \quad y \in (-a, a),$$

with radius of convergence a. Switch back to $x = a + y$, we have

$$\ln x = \ln a + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{x-a}{a}\right)^n, \quad |x - a| < a.$$

Since a is arbitrary, for each $a \in (0, \infty)$, there is a neighborhood of x such that $\ln x$ is represented by a convergent power series. Hence, $\ln x$ is real analytic on $(0, \infty)$.

Exercise 5 (Tao II Exercise 4.5.7, 10 points). Let $f : (0, \infty) \to \mathbb{R}$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) = Ce^x$ for some positive constant C; justify your reasoning. *(Hint: there are basically three different proofs available. One proof uses the logarithm function, another proof uses the function e^{-x}, and a third proof uses power series. Of course, you only need to supply one proof.)*

Solution 5. Since $f(x)$ is analytic, it is infinitely differentiable and given exactly by its Taylor series at any $x \in \mathbb{R}$. Since $f'(x) = f(x)$, by induction $f^{(n)}(x) = f(x)$ for any $n \geq 1$. Fix some $a > 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f(a)}{n!} x^n = f(a)e^x, \quad f(a) \in \mathbb{R}_{>0} \text{ is a constant.}$$

Exercise 6 (Tao II Exercise 4.5.8, 10 points). Let $m > 0$ be an integer. Prove

$$\lim_{x \to +\infty} \frac{e^x}{x^m} = +\infty.$$

without using the L'Hopital's rule. Hint: $e^x \geq \sum_{k=0}^{m+1} \frac{x^k}{k!}$ for $x > 0$.

Solution 6. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and each term in the series is nonnegative when $x > 0$, we have $e^x \geq \sum_{n=0}^{m+1} \frac{x^n}{n!}$ for $x > 0$. Then, for any $N > 0$

$$\frac{e^x}{x^m} = \sum_{n=0}^{\infty} \frac{x^{n-m}}{n!} \geq \sum_{n=0}^{m+1} \frac{x^{n-m}}{n!} > \frac{x}{(m+1)!} > \frac{N}{(m+1)!}$$

whenever $x > N$. Therefore, $\lim_{x \to +\infty} \frac{e^x}{x^m} = +\infty$.

Exercise 7 (Tao II Exercise 4.5.9, 10 points). Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^x > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate c, will eventually overtake any given polynomial $P(x)$, no matter how large. Hint: use Exercise 4.5.8.

Solution 7. Let $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ be a polynomial of degree n. Then, for any $x > 0$, we have

$$|P(x)| \leq \sum_{k=0}^{n} |a_k| x^k \leq M x^n,$$

where $M = |a_n| + |a_{n-1}| + \cdots + |a_0|$. From Exercise 4.5.8, we know that $\lim_{x \to +\infty} \frac{e^{cx}}{x^n} = +\infty$. Therefore, there exists some $N > 0$ such that for all $x > N$, we have

$$\frac{e^{cx}}{x^n} > M \implies e^{cx} > M x^n \geq |P(x)|.$$

Thus, we conclude that there exists some real number $N > 0$ such that $e^{cx} > |P(x)|$ for all $x > N$.

物理三 黃紹凱 B12202004
November 21, 2025

You can do the following problems to practice.
You don't have to submit the following problems.

Exercise 8 (Tao II Exercise 4.5.4, Optional). Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $f(x) := \exp(-1/x)$ when $x > 0$, and $f(x) := 0$ when $x \le 0$. Prove that f is infinitely differentiable, and $f^{(k)}(0) = 0$ for every integer $k \ge 0$, but that f is not real analytic at 0.

Solution 8. Since both 0 and $e^{-1/x}$ are compositions of elementary functions, they are infinitely differentiable on their respective domains. We only need to show that f is infinitely differentiable at $x = 0$ and $f^{(k)}(0) = 0$ for all $k \ge 0$.

Claim. For $x > 0$, the n-th derivative of $e^{-1/x}$ is

$$f^{(n)}(x) = e^{-1/x}(-1)^n \sum_{k=1}^{n} \binom{n+k}{n-k}\binom{n+k-1}{n-k}(n-k)!\,(-1)^k x^{-(n+k)}.$$

Proof. For $n = 1$, $f'(x) = \frac{1}{x^2}e^{-1/x}$, so the base case is satisfied. Suppose the formula holds for n, then for $n+1$, we have

$$\begin{aligned}
f^{(n+1)}(x) &= \frac{d}{dx} f^{(n)}(x) \\
&= \frac{d}{dx}\left(e^{-1/x}(-1)^n \sum_{k=1}^{n} \binom{n+k}{n-k}\binom{n+k-1}{n-k}(n-k)!\,(-1)^k x^{-(n+k)}\right) \\
&= e^{-1/x}(-1)^{n+1} \sum_{k=1}^{n} \frac{(n+k)!}{(n-k)!(2k)!}\frac{(n+k-1)!}{(n-k)!(2k-1)!}(n-k)!(n+k)\,(-1)^k x^{-(n+1+k)} \\
&= e^{-1/x}(-1)^{n+1} \sum_{k=1}^{n+1} \binom{n+1+k}{n+1-k}\binom{n+k}{n+1-k}(n+1-k)!\,(-1)^k x^{-(n+1+k)}.
\end{aligned}$$

Thus, the formula holds for all $n \ge 1$ by induction. □

By our claim, for any $n \ge 1$, we have

$$\begin{aligned}
\lim_{x\to 0^+} f^{(n)}(x) &= \lim_{x\to 0^+} e^{-1/x}(-1)^n \sum_{k=1}^{n} \binom{n+k}{n-k}\binom{n+k-1}{n-k}(n-k)!\,(-1)^k x^{-(n+k)} \\
&= \lim_{u\to +\infty} e^{-u}(-1)^n \sum_{k=1}^{n} \binom{n+k}{n-k}\binom{n+k-1}{n-k}(n-k)!\,(-1)^k u^{n+k} \\
&= 0,
\end{aligned}$$

by Exercise 4.5.8, while $\lim_{x\to 0^-} f^{(n)} = 0$. Therefore, $f^{(n)}(0) = 0$ for all $n \ge 0$ and f is differentiable. Since the Taylor series of f at 0 is identically zero, but $f(x) > 0$ for all $x > 0$, f is not real analytic at 0.

Exercise 9 (Optional). In class, we proved that the function $f(x) = a^x$ is continuous on $\mathbb{Q}$ for $a > 1$. Let $n \in \mathbb{N}$. Prove that f is uniformly continuous on the rational interval

$$[-n, n] \cap \mathbb{Q}.$$

Remark. If this is true, then $f(x) = a^x$ admits a unique continuous extension to all real numbers $x \in [-n, n]$.

Solution 9. Since f is continuous on $\mathbb{Q}$, for any $\epsilon > 0$ and $x \in [-n,n] \cap \mathbb{Q}$, there exists some $\delta_x > 0$ such that for any $y \in \mathbb{Q}$ with $|x-y| < \delta_x$, we have $|f(x) - f(y)| < \epsilon$. The collection of open intervals $\{(x - \delta_x/2, x + \delta_x/2) : x \in [-n,n] \cap \mathbb{Q}\}$ forms an open cover of the compact set $[-n,n]$. Thus, there exists a finite subcover $\{(x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2) : i = 1, 2, \ldots, m\}$. Let $\delta = \min_{1 \le i \le m} \delta_{x_i}/2 > 0$. Then, for any $x, y \in [-n,n] \cap \mathbb{Q}$ with $|x - y| < \delta$, there exists some i such that $x \in (x_i - \delta_{x_i}/2,\ x_i + \delta_{x_i}/2)$. Therefore,

$$|y - x_i| \le |y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2} \le \delta_{x_i}.$$

Hence, by the triangle inequality, we have

$$|f(x) - f(y)| \le |f(x) - f(x_i)| + |f(x_i) - f(y)| < \epsilon + \epsilon = 2\epsilon,$$

and f is uniformly continuous on $[-n,n] \cap \mathbb{Q}$.

Exercise 10 (Optional)**.** Define the sequence

$$\forall n \ge 1, \qquad S_n = \sum_{k=1}^{n} \ln k.$$

(a) Show that for every $k \ge 2$, we have

$$\int_{k-1}^{k} \ln t \, dt \ \le \ \ln k \ \le \ \int_{k}^{k+1} \ln t \, dt.$$

Deduce that

$$S_n = n \ln n - n + o(n).$$

(b) By considering the sequence $(A_n)_{n \ge 1}$, defined by

$$\forall n \ge 1, \qquad A_n = S_n - n \ln n + n,$$

show that $A_n - A_{n-1} \sim \dfrac{1}{2n}$ and deduce that

$$A_n \sim \frac{1}{2} \ln n.$$

(c) Let

$$D_n := S_n - n \ln n + n - \frac{1}{2} \ln n \qquad \text{for } n \ge 1.$$

Show that

$$D_n - D_{n-1} \sim -\frac{1}{12n^2}.$$

(d) Show that D_n converges to some D_∞ when $n \to \infty$. Deduce that there exists some constant $C > 0$ such that

$$n! \sim C \left(\frac{n}{e}\right)^n \sqrt{n}.$$

(e) Using the expression of $I_{2n} = \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{\frac{\pi}{4n}}\,(1 + o(1))$ (proved in the following), show that

$$C = \sqrt{2\pi}.$$

(f) Show that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right).$$

Solution 10.

(a) For $k \geq 2$, since $\ln t$ is increasing on $(0,\infty)$, we have

$$\int_{k-1}^{k} \mathrm{d}t\ \ln t \leq \int_{k-1}^{k} \mathrm{d}t\ \ln k = \ln k \leq \int_{k}^{k+1} \mathrm{d}t\ \ln t.$$

Summing over $k = 2, 3, \ldots, n$, we have

$$\int_{1}^{n} \mathrm{d}t\ \ln t \leq S_n \leq \int_{2}^{n+1} \mathrm{d}t\ \ln t$$

and hence

$$n \ln n - n + 1 \leq S_n \leq (n+1)\ln(n+1) - (n+1) + 1 \implies S_n = n\ln n - n + o(n).$$

(b) We have

$$\begin{aligned}
A_n - A_{n-1} &= S_n - S_{n-1} - n\ln n + n + (n-1)\ln(n-1) - (n-1) \\
&= \ln n - n\ln n + (n-1)\ln(n-1) + 1 \\
&= 1 + (n-1)\ln\left(1 - \frac{1}{n}\right) \\
&= 1 + (n-1)\left(-\frac{1}{n} - \frac{1}{2n^2} + o\Big(\frac{1}{n^2}\Big)\right) = \frac{1}{2n} + R(n),
\end{aligned}$$

where $\lim_{n\to\infty} \frac{R(n)}{n^{-3}} = 0$. Then we have

$$\lim_{n\to\infty} \frac{A_n - A_{n-1}}{\frac{1}{2n}} = 1 + 2nR(n) = 1 \implies A_n - A_{n-1} \sim \frac{1}{2n}.$$

(c) We have

$$\begin{aligned}
D_n - D_{n-1} &= \left(S_n - n\ln n + n - \frac{1}{2}\ln n\right) - \left(S_{n-1} - (n-1)\ln(n-1) + (n-1) - \frac{1}{2}\ln(n-1)\right) \\
&= \ln n - \ln(n-1) + n\ln\left(1 - \frac{1}{n}\right) + 1 + \frac{1}{2}\ln\left(1 - \frac{1}{n}\right) \\
&= 1 + (n - \frac{1}{2})\ln\left(1 - \frac{1}{n}\right) \\
&= 1 + \left(n - \frac{1}{2}\right)\left[-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)\right] = -\frac{1}{12n^2} + R(n),
\end{aligned}$$

where $\lim_{n\to\infty} \frac{R(n)}{n^{-4}} = 0$. Then we have

$$\lim_{n\to\infty} \frac{D_n - D_{n-1}}{\frac{1}{12n^2}} = 1 - 12n^2R(n) = 1 \implies D_n - D_{n-1} \sim -\frac{1}{12n^2}$$

(d) Let $G_n = D_n - D_{n-1}$. Since $D_n - D_{n-1} \sim -\frac{1}{12n^2}$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left|12n^2 G_n + 1\right| < \varepsilon$ whenever $n > N$. Then,

$$|G_n| < \left|G_n + \frac{1}{12n^2}\right| < \frac{\varepsilon}{12n^2} < \varepsilon, \quad \text{whenever } n > N.$$

Hence, $\{D_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and by completeness of the reals there is a unique limit D_∞ in $\mathbb{R}$.By definition of D_n, we have

$$S_n = \ln n! = n\ln n - n + \frac{1}{2}\ln n + D_n.$$

Exponentiating both sides gives $n! = e^{D_n}\sqrt{n}\left(\frac{n}{e}\right)^n$, and

$$\lim_{n\to\infty} \frac{n!}{\sqrt{n}\left(\frac{n}{e}\right)^n} = \lim_{n\to\infty} e^{D_n} = e^{D_\infty} \equiv C > 0 \implies n! \sim C\left(\frac{n}{e}\right)^n \sqrt{n}.$$

(e) From the expression of I_{2n}, we have

$$I_{2n} = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{\frac{\pi}{4n}}(1 + o(1)).$$

Using the identity from part (d), we have

$$I_{2n} = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{\pi}{2} \cdot \frac{C\left(\frac{2n}{e}\right)^{2n}\sqrt{2n}}{2^{2n}C^2\left(\frac{n}{e}\right)^{2n} n} = \frac{\pi}{2C}\sqrt{\frac{2}{n}}.$$

Therefore, we have $\frac{\pi}{2C}\sqrt{\frac{2}{n}} = \sqrt{\frac{\pi}{4n}}$, and hence $C = \sqrt{2\pi}$.

(f) From part (e), we have $D_\infty = \log C = \frac{1}{2}\log(2\pi)$. Then,

$$\begin{aligned} D_n - D_\infty &= S_n - n\ln n + n - \frac{1}{2}\ln n - D_\infty \\ &= \ln n! - n\ln n + n - \frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) \\ &= \ln\left(\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right). \end{aligned}$$

From part (c), we have

$$D_n - D_\infty \sim -\frac{1}{12n^2} \implies \lim_{n\to\infty} \frac{D_n - D_\infty}{-\frac{1}{12n^2}} = 1.$$

Therefore, for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\left|\frac{D_n - D_\infty}{-\frac{1}{12n^2}} - 1\right| < \varepsilon \implies \left|D_n - D_\infty + \frac{1}{12n^2}\right| < \frac{\varepsilon}{12n^2}$$

whenever $n > N$. Hence, we have

$$\left|\ln\left(\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right) + \frac{1}{12n^2}\right| < \frac{\varepsilon}{12n^2} \implies \left|\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} - e^{-\frac{1}{12n^2}}\right| < e^{-\frac{1}{12n^2}}\left(e^{\frac{\varepsilon}{12n^2}} - 1\right)$$

whenever $n > N$. Since $e^x = 1 + x + o(x)$ as $x \to 0$, we have

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = e^{-\frac{1}{12n^2}} + o\left(\frac{1}{n^2}\right) = 1 - \frac{1}{12n^2} + o\left(\frac{1}{n^2}\right).$$

Exponentiating both sides gives

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right).$$

Exercise 11 (Optional)**.** Let $\mathcal{P}$ be the set of all the primes. In this exercise, we will prove that $\sum_{p\in\mathcal{P}} \frac{1}{p}$ is divergent.

(a) Show that for $s > 1$, we have

$$-\sum_{p\in\mathcal{P}} \log\left(1 - \frac{1}{p^s}\right) = \log\zeta(s).$$

(b) Deduce that there exists $M > 0$ such that for any $s > 1$, we have

$$\left|\sum_{p\in\mathcal{P}} \frac{1}{p^s} - \log\zeta(s)\right| < M.$$

(c) Show that as $s \to 1^+$, we have $\zeta(s) \to +\infty$.

(d) Conclude that

$$\sum_{p \in \mathcal{P}} \frac{1}{p}$$

is divergent.

Solution 11.

(a) The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Fix $s > 1$ and $N \in \mathbb{N}$. Consider the finite product

$$P_N \equiv \prod_{p \in \mathcal{P}, p \leq N} \frac{1}{1 - p^{-s}} = \prod_{p \in \mathcal{P}, p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \sum_{n \in A} \frac{1}{n^s},$$

where A is the set of numbers all of whose prime factors are less than N. Let S_N be the N-th partial sum of $\zeta(s)$, then since P_N contains all terms of the form $\frac{1}{n^s}$ for $n \leq N$ by the Fundamental Theorem of Arithmetic, $S_N \leq P_N$. On the other hand, $P_N \leq \zeta(s)$ since it is a sum of a subsequence of terms in $\zeta(s)$, which are all positive. Therefore, we have $S_N \leq P_N \leq \zeta(s)$ for all $N \in \mathbb{N}$. $S_N \to \zeta(s)$ as $N \to \infty$ by definition, and P_N is an increasing sequence in N, so by the Squeeze Theorem $P_N \to \zeta(s)$ as $N \to \infty$. Hence, we have

$$\zeta(s) = \lim_{N \to \infty} P_N = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} \implies -\sum_{p \in \mathcal{P}} \log\left(1 - \frac{1}{p^s}\right) = \log \zeta(s).$$

(b) Using the Taylor expansion of $\log(1-x)$, we have

$$-\log\left(1 - \frac{1}{p^s}\right) = \frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots.$$

Since $p > 1$, $p^{-ms} < 1$ for all $m > 0$, so the Taylor series always converges. Therefore,

$$\log \zeta(s) = -\sum_{p \in \mathcal{P}} \log\left(1 - \frac{1}{p^s}\right) = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{kp^{ks}},$$

and hence

$$\left|\sum_{p \in \mathcal{P}} \frac{1}{p^s} - \log \zeta(s)\right| = \left|\sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{kp^{ks}}\right| \leq \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{p^{ks}} = \sum_{p \in \mathcal{P}} \frac{1}{p^{2s}(1 - p^{-s})}.$$

Since $s > 1$, $1 - p^{-s} \leq 1 - 2^{-s} < \frac{1}{2}$, so

$$\left|\sum_{p \in \mathcal{P}} \frac{1}{p^s} - \log \zeta(s)\right| \leq 2 \sum_{p \in \mathcal{P}} \frac{1}{p^{2s}} < 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \equiv M < \infty.$$

The series converges by comparison test with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(c) Near $s = 1$, uniform convergence fails so we cannot switch the order of limit and summation. For $s > 1$, consider $f(x) = x^{-s}$, which is positive decreasing on $[1, \infty]$. Then, by the integral test, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \int_1^{\infty} \mathrm{d}x\, x^{-s} = \frac{1}{s-1}.$$

Therefore, as $s \to 1^+$, $\zeta(s) \to +\infty$.

(d) Suppose $\sum_{p\in\mathcal{P}} \frac{1}{p}$ converges, then $\lim_{s\to 1^+} \sum_{p\in\mathcal{P}} \frac{1}{p^s}$ is bounded. Then, by (b),

$$\lim_{s\to 1^+} \log\zeta(s) \text{ is bounded} \implies \lim_{s\to 1^+} \zeta(s) \text{ is bounded},$$

a contradiction to (c). Therefore, $\sum_{p\in\mathcal{P}} \frac{1}{p}$ diverges.

Exercise 12 (Optional)**.**

Theorem 1 (Wallis Integrals — Factorial Version)**.** For each integer $n \geq 0$, define

$$I_n := \int_0^{\pi/2} \sin^n x\, dx.$$

Then:

(a)

$$I_0 = \frac{\pi}{2}, \qquad I_1 = 1.$$

(b) For all $n \geq 2$,

$$nI_n = (n-1)I_{n-2}.$$

(c) For each $m \in \mathbb{N}$,

$$I_{2m-1} = \frac{2^{\,2m-1}(m-1)!\,m!}{(2m)!}, \qquad I_{2m} = \frac{\pi}{2}\cdot\frac{(2m)!}{2^{\,2m}(m!)^2}.$$

(d) For all $n \geq 1$,

$$I_n I_{n-1} = \frac{\pi}{2n}.$$

(e) As $n \to \infty$,

$$I_n = \sqrt{\frac{\pi}{2n}}\,(1 + o(1)).$$

(f) In particular,

$$I_{2n} = \frac{\pi}{2}\cdot\frac{(2n)!}{2^{\,2n}(n!)^2}.$$

Solution 12. Here we provide a proof for Theorem 1.

(a) By directly computing the integrals, we have

$$I_0 = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}, \qquad I_1 = \int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2} = 1.$$

(b) For $n \geq 2$, we have

$$I_n = \int_0^{\pi/2} \mathrm{d}x\, \sin^n x = \int_0^{\pi/2} \mathrm{d}x\, \sin^{n-1} x \sin x.$$

Do integration by parts with $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, we have $v = -\cos x$, $\mathrm{d}u = (n-1)\sin^{n-2} x\cos x\,\mathrm{d}x$. Then,

$$\begin{aligned} I_n &= \left[-\sin^{n-1} x\cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \mathrm{d}x\, \sin^{n-2} x\cos^2 x \\ &= (n-1)\int_0^{\pi/2} \mathrm{d}x\, \sin^{n-2} x(1-\sin^2 x) = (n-1)I_{n-2}. \end{aligned}$$

(c) We should discuss the two cases where n is an odd or even integer, with I_0 and I_1 from part (a) as the base cases. For $n = 2m-1$, where $m \in \mathbb{N}$, we have

$$I_{2m-1} = \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{2}{3} \cdot I_1 = \frac{2^{\,2m-1}(m-1)!\, m!}{(2m)!}.$$

On the other hand, for $n = 2m$, we have

$$I_{2m} == \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot I_0 = \frac{\pi}{2} \cdot \frac{(2m)!}{2^{\,2m}(m!)^2}.$$

(d) Again, we discuss the cases when n is an even or an odd number. For $n = 2m+1$,

$$I_{2m+1} I_{2m} = \frac{\pi}{2(2m+1)} = \frac{\pi}{2n}.$$

Thus,

$$I_n I_{n-1} = \frac{\pi}{2n}.$$

For $n = 2m$, a similar calculation gives

$$I_{2m} I_{2m-1} = \left(\frac{\pi}{2} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \right) \left(\frac{2^{2m-1}(m-1)!\, m!}{(2m)!} \right) = \frac{\pi}{2(2m)} = \frac{\pi}{2n}.$$

Hence, $I_n I_{n-1} = \frac{\pi}{2n}$, for all $n \geq 1$.

(e) Notice that since $\sin x \in [0,1]$ for all $x \in [0, \pi/2]$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx \leq \int_0^{\pi/2} \sin^{n-1} x \, dx = I_{n-1},$$

and hence $\{I_n\}_{n=0}^{\infty}$ is a positive, decreasing sequence. From the product identity in part (d) and $I_{n-1} \leq I_n \leq I_{n+1}$, we have

$$\frac{\pi}{2(n+1)} = I_{n+1} I_n \leq I_n^2 \leq I_n I_{n-1} = \frac{\pi}{2n}.$$

Everything is positive, so, multiplying by $\frac{2n}{\pi}$ and taking square roots, we get

$$\sqrt{\frac{n}{n+1}} \leq \sqrt{\frac{2n}{\pi}}\, I_n \leq 1 \implies I_n \sim \sqrt{\frac{\pi}{2n}},$$

or, equivalently, using the little-o notation gives

$$I_n = \sqrt{\frac{\pi}{2n}}\, (1 + o(1))$$

(f) Directly from part (c), we have

$$I_{2n} = \frac{\pi}{2} \cdot \frac{(2n)!}{2^{\,2n}(n!)^2}.$$