# Limit Cycles in Two Dimensions

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Firefly synchronization can be understood using the Kuramoto model, an example of a nonlinear dynamical system.

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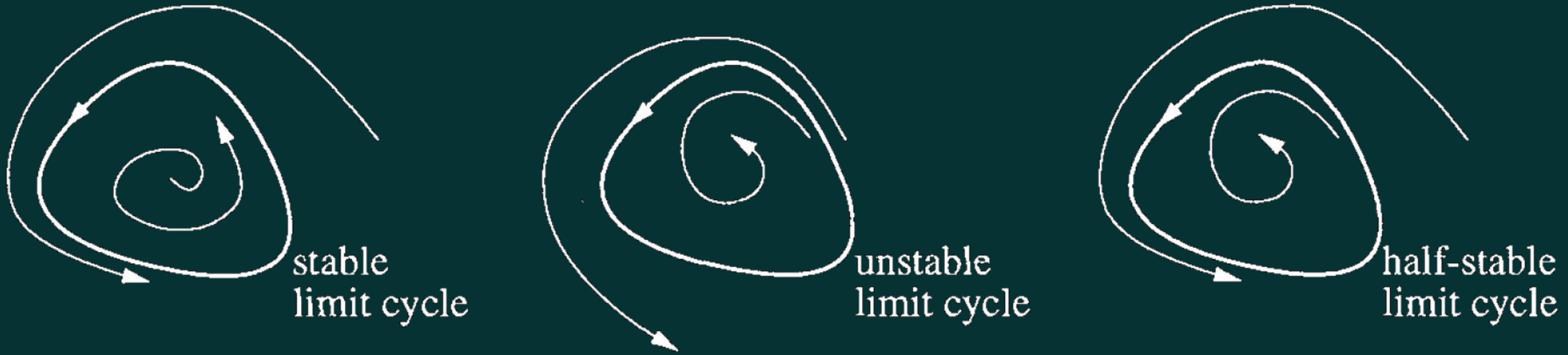
# About The Lecture

- Dynamical system is the study of ODEs
- Nonlinear phenomena
- Living systems are inherently dynamical
- 通靈 is very important
- These slides have too many words

# What Is A Limit Cycle?

**Def. (limit cycle 極限環)**

An <u>isolated</u> <u>closed</u> trajectory. Limit cycles arise in minimally two-dimension systems.




*Strogatz* Fig. 7.0.1

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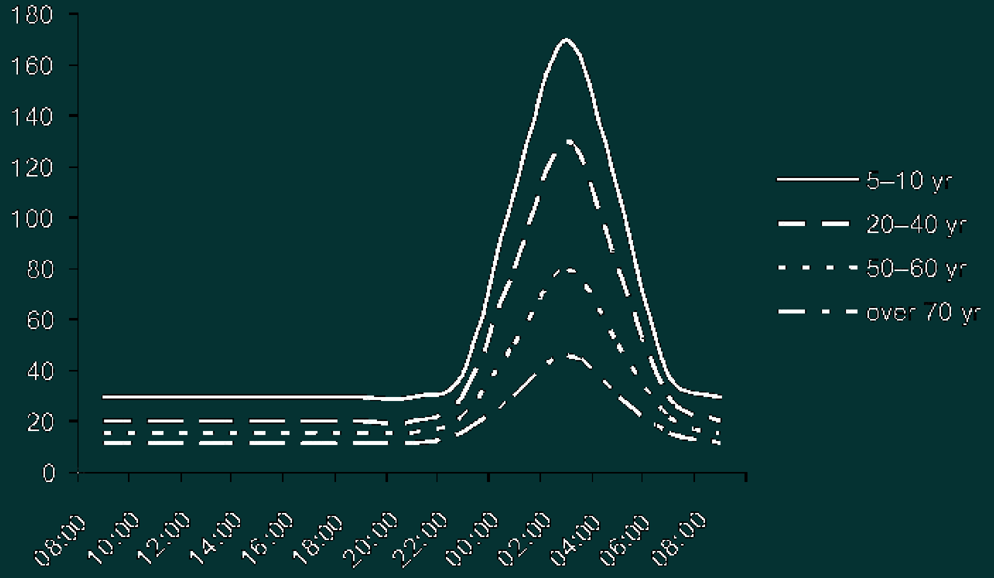
# But Why?

*Given a 2D system, does it have any periodic solutions? What about chaos?*

# But Why?

- Beating of a heart
- Body temperature
- Circadian cycle
- Oscillatory chemical reactions
- Self-excited vibrations in bridges, e.g. Tacoma bridge incident

Melatonin level through day

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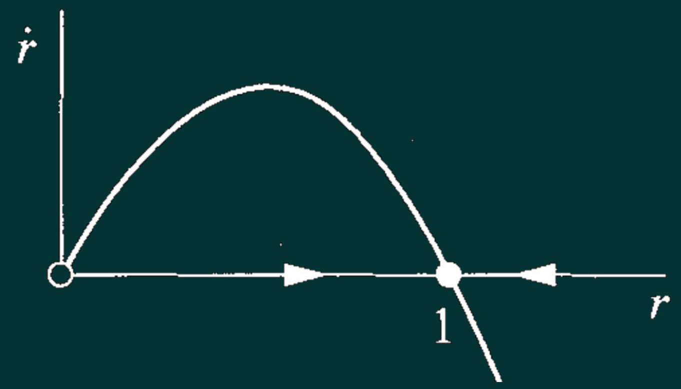
# Two Examples

**(I) Polar coordinates**

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

$r^* = 0$ is unstable; $r^* = 1$ is stable.

Trajectories approach a stable limit cycle (**Q:** what is the limit cycle?).

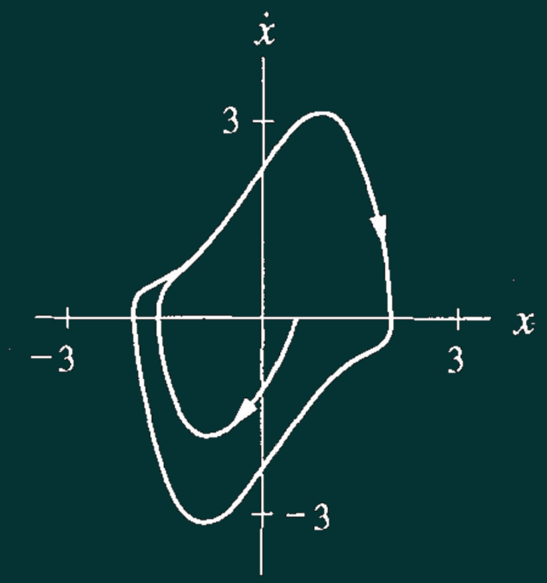



*Strogatz* Fig. 7.1.1

**(II) Van der Pol (VdP) oscillator**

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

**Thm.** VdP has a unique, stable limit cycle for each $\mu \geq 0$.




*Strogatz* Fig. 7.1.4

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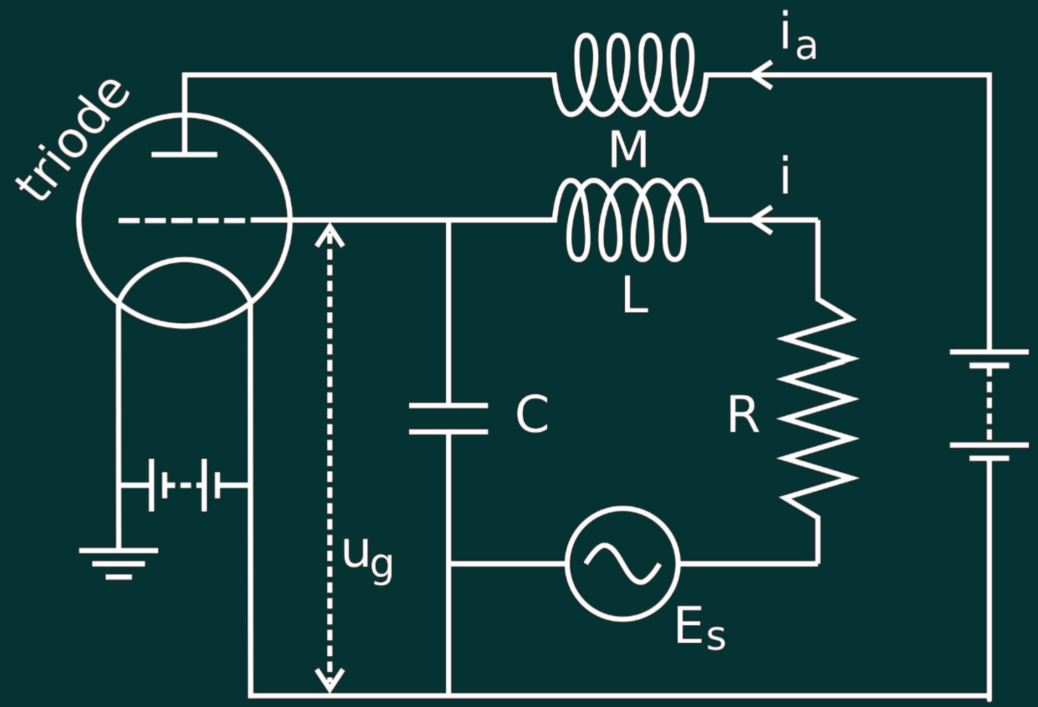
# Nonlinear Circuit With A Triode








Adapted from Wikimedia Commons

# Gradient Systems Have No Closed Orbits

**Def. (gradient system 梯度系統)**

A system is called a gradient system (GS) if it can be written as $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$ for some potential function $V(\mathbf{x}) \in C^1$.

**Thm.** Closed orbits are impossible in GSs.

**Ex.** Prove the above thm.

**Ex.** Limit cycles are not possible in 1D. Show that, in fact, all 1D systems are gradient systems.

**pf.** Consider the change in $V(\mathbf{x})$ after one cycle:

$$\Delta V = \int_0^T \mathrm{d}t \, \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \int_0^T \mathrm{d}t \, (\nabla V \cdot \dot{\mathbf{x}})$$

$$= -\int_0^T \mathrm{d}t \, \|\mathbf{x}\|^2 < 0$$

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# What If It Is NOT A GS?

**(I) Gradient System**

$\dot{x} = \sin y\,,\ \ \dot{y} = x\cos y$ has no closed orbits, since it has a potential function $V(x, y) = -x\sin y$.

**(II) NOT Gradient System**

The nonlinear damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no closed orbits, since the energy $E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$ has $\Delta E < 0$ after one cycle.

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# INTERMISSION: Stability of Dynamical Systems

**(I) Lyapunov Stability**

For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t$:

$$\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta \Longrightarrow \|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon.$$

**(II) Asymptotic Stability**

Lyapunov stable, AND there exists $\delta > 0$ such that

$$\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta \Longrightarrow \lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0.$$

**(III) Global Asymptotic Stability**

Lyapunov stable, AND $\lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0$ for all $\mathbf{x}$.

# Lyapunov Function To The Rescue!

*Construct an* ***ENERGY FUNCTION***

**Def. (Lyapunov function)**

Suppose $\dot{\mathbf{x}} = F(\mathbf{x})$ has fixed point $\mathbf{x}^*$. A map $V \in C^1$ is a positive-definite, i.e. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$ is called a Lyapunov function.

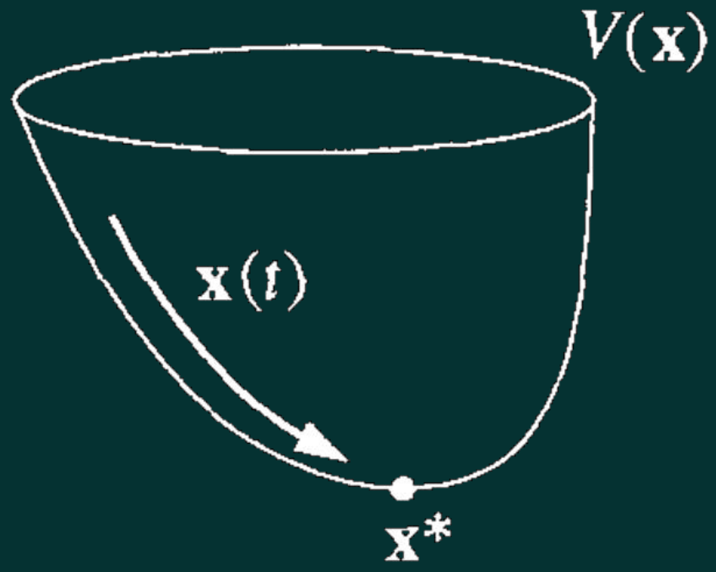
# Lyapunov Function To The Rescue!

$\dot{V}(\mathbf{x}^*)$ Determines stability of $\mathbf{x}^*$

(1) Stability $\dot{V}(\mathbf{x}^*) \leq 0$

(2) Global Asymptotic Stability (GAS)

$\dot{V}(\mathbf{x}^*) < 0$ (with one fixed point)




*Strogatz* Fig. 7.2.1

**Ex.** We showed that $\ddot{x} + (\dot{x})^3 + x = 0$ has no closed orbits. Find a Lyapunov function for it.

**Sol.** Let $V(x, y) = (x + y)^2 + y^2 + x^4$.

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# Global Asymptotic Stability

**Thm. (Barbashin-Krasovskii-LaSalle)**

Suppose $\dot{\mathbf{x}} = F(\mathbf{x})$ has a fixed point $\mathbf{x}^*$. If there exists a map $V(\mathbf{x}) \in C^1$ such that

(1) $\dot{V}$ is negative-semidefinite ($\dot{V}(\mathbf{x}) \leq 0$).

(2) $V$ is positive-definite

and the invariance set $I = \left\{x \mid \dot{V}(x) = 0\right\}$ does not contain any trajectory except $\mathbf{x}(t) = \mathbf{0}$, then $\mathbf{0}$ is AS. If $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then $0$ is GAS.

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# Applying the Theorem is Tough!

**(I) Cute example** [Chris Grant, *LaSalle's Invariance Principle*]

Show that $\mathbf{0}$ is GAS for $\dot{x} = -y - x^3, \dot{y} = x^5$.

**Sol.** Let $V(x, y) = x^6 + 3y^2$.

# Applying the Theorem is Tough!

**(II) Pendulum with friction is LAS**

*Intuition:* a pendulum with friction eventually stops.

$$ml\ddot{\theta} + mg\sin\theta + \mu l\dot{\theta} = 0$$

**pf.** Let $\dot{u} = v,\ \dot{v} = -\frac{g}{l}\sin u - \frac{\mu}{m}v$. Motivated by energy,

define $V(u, v) = \frac{g}{l}(1 - \cos u) + \frac{1}{2}v^2$.

Check: $V(0) = 0,\ \dot{V}(0) = 0,\ \dot{V}(u, v) \leq 0$, and the invariance set is the trivial singleton $I = \{0\}$. So apply LaSalle's theorem.

# No Orbits: Bendixson-Dulac Theorem

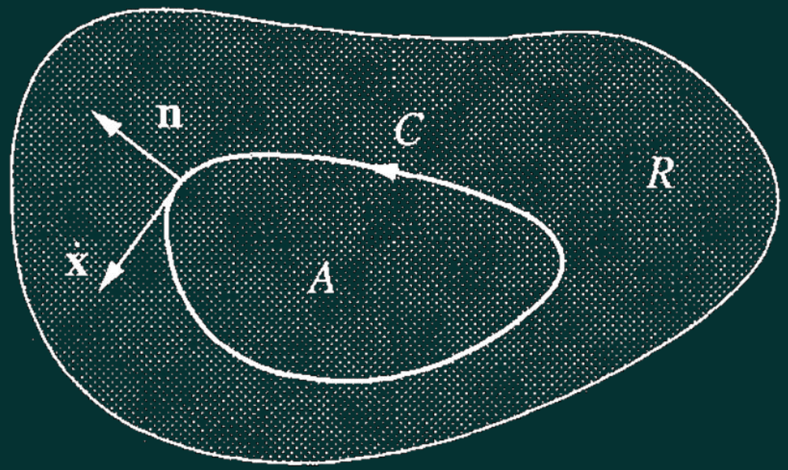
**Thm. (Bendixson-Dulac)**

Let $\mathbf{x} = F(\mathbf{x})$ be a $C^1$ vector field on a simply connected subset $R$ of the plane. If there exists $g(\mathbf{x}) \in C^1$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ does not change sign throughout $R$ , then there are <u>no closed orbits entirely in $R$</u> .

**pf.** Suppose $C \subseteq R$ is a closed orbit. But

$$0 \neq \iint \mathrm{d}A\, \nabla \cdot (g\dot{\mathbf{x}}) = \oint \mathrm{d}l\, (\mathbf{n} \cdot \dot{\mathbf{x}}) g = 0,$$

By Green's Theorem (格林定理).




*Strogatz* Fig. 7.2.2

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# Exercises -- When Is A Closed Orbit Unique?

**Ex.** (*Strogatz* 7.2.17)
Assume the hypotheses of Dulac's criterion, except now suppose that $R$ is <u>topologically equivalent to an annulus</u>. Using Green's theorem, show that there exists at most one closed orbit in R. (This result can be useful sometimes as a way of proving that a closed orbit is unique.)

**Ex.** (*Strogatz* 7.3.10)
Consider the two-dimensional system $\dot{x} = Ax - \|x\|^2 x$, where $A \in M_{2\times 2}(\mathbb{R})$ has complex eigenvalues $\alpha + \mathrm{i}\omega$. Prove that there exists at !east one limit cycle for $\alpha > 0$ and that there are none for $\alpha < 0$.

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# Yes Orbits: Poincare-Bendixson Theorem

**KEY RESULT**

**Thm. (Poincare-Bendixson)**

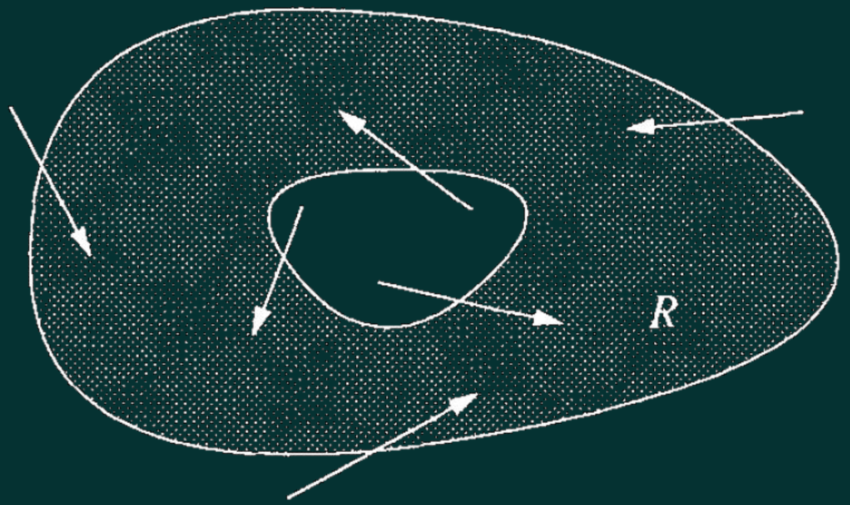
Suppose that

(1) $R \subseteq \mathbb{R}^2$ is closed and bounded.

(2) $\dot{\mathbf{x}} = F(\mathbf{x})$ is continuously differentiable on an open set $U \supseteq R$.

(3) $R$ does not contain fixed points.

(4) There is a trajectory $C$ in $R$ that never leaves $R$. Then $R$ contains a closed orbit.




*Strogatz* Fig. 7.3.2

Construct a **trapping region** (vector field points "inward" everywhere on the boundary)

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# When Perturbation Preserves Limit Cycles

**Q:** The system $\dot{r} = r(1 - r^2),\ \dot{\theta} = 1$ has a limit cycle. What about the pertubed system?

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta\,, \quad \dot{\theta} = 1$$

**A:** There is a closed orbit somewhere in the annulus $0.9999\sqrt{1-\mu} < r < 1.0001\sqrt{1+\mu}$, when $\mu$ is small enough.

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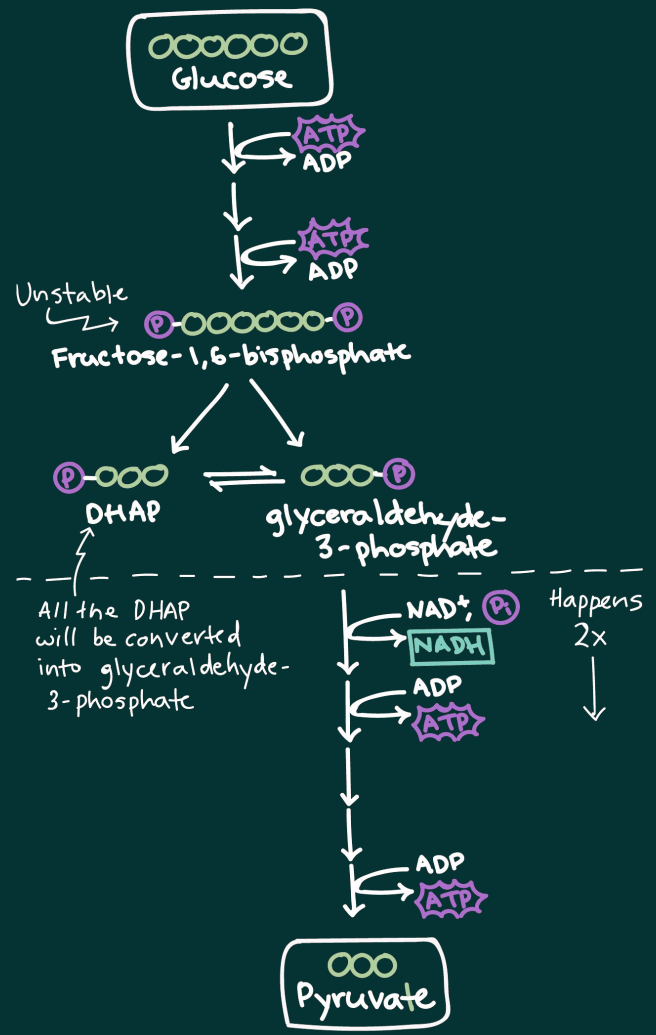
# Oscillatory Processes Are Everywhere In Biology

In intact yeast cells, glycolysis (糖解反應) can proceed in an oscillatory fashion.

**Selkov model**: $\dot{x} = -x + ay + x^2 y, \ \ \dot{y} = b - ay - x^2 y$, where $x = [\mathrm{ADP}]$ and $y = [\mathrm{F6P}]$.

**Ex.** Show there is a limit cycle (and hence oscillation) by constructing a trapping region.




Credit: Khan Academy

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# Oscillatory Processes in Biology

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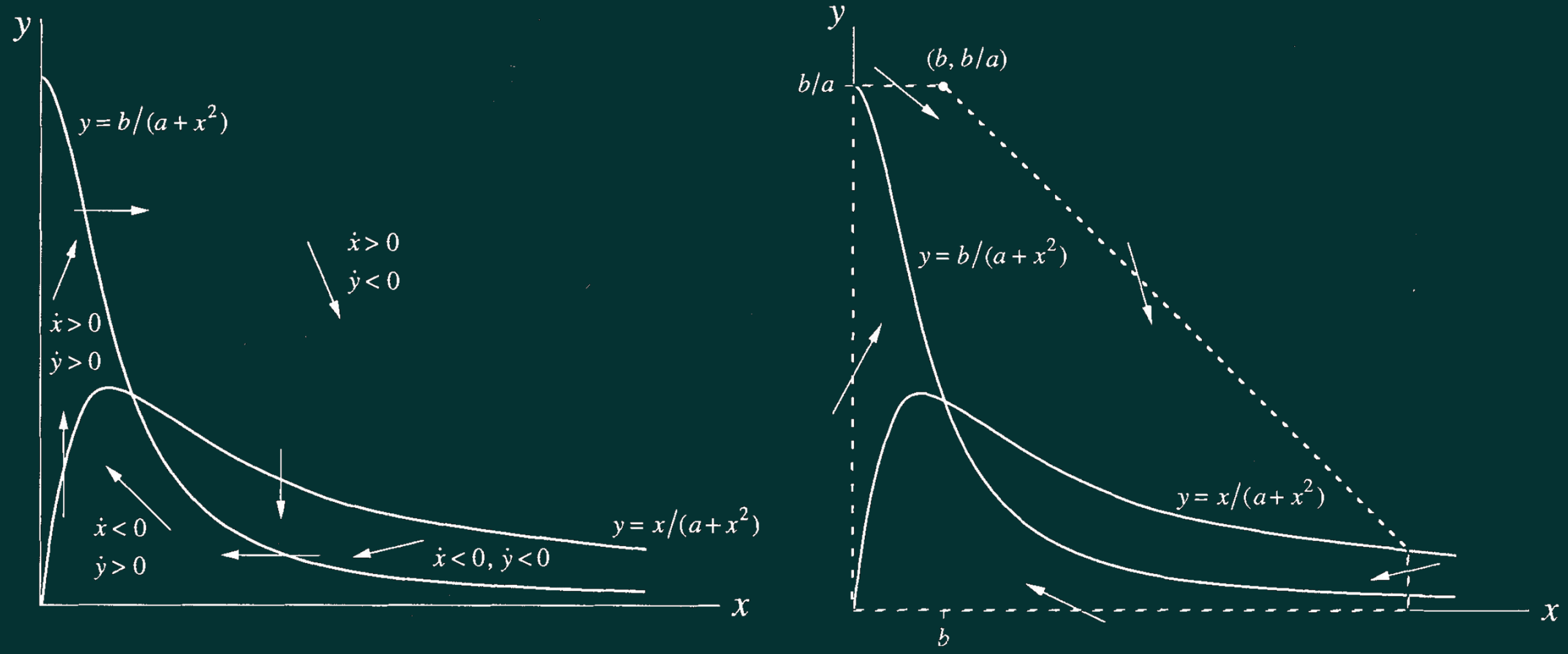


*Strogatz* Fig. 7.3.4 / 7.3.5

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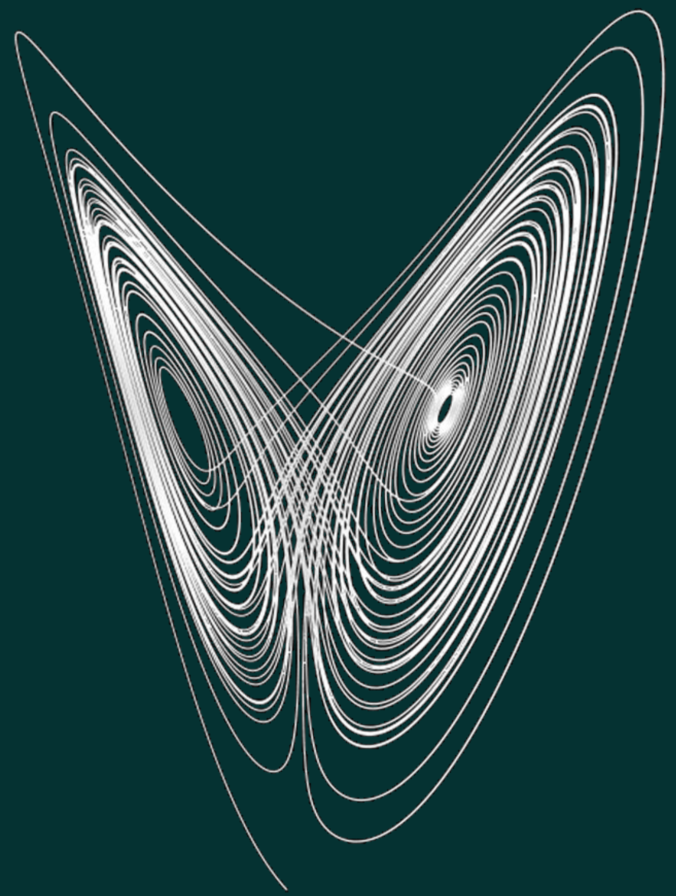
# Where and How Does Chaos (Not) Occur

**Cor. (no chaos in 2D)**

In a 2D $C^1$ system, a bounded trajectory can only approach a (1) fixed point or a (2) closed periodic orbit.

**Q:** What is the difference between a closed periodic orbit and a limit cycle?

Chaos needs $\geq$ 3D, e.g. Lorenz attractor.



Credit: Paul Bourke, Lorenz attractor

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# Lienard System Has A Unique Stable Limit Cycle

**Def. (Lienard's equations)**

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

Generalizes VdP.

*Interpretation:*

(2)(3) g acts like a spring

(4)(5) Damping by f is negative at small $|x|$ and positive at large $|x|$.

**Thm. (Lienard)**

Suppose (1) $f(x), g(x) \in C^1$; (2) $g$ is odd; (3) $g(x) > 0$ for $x > 0$; (4) $f$ is even; (5) The function $F(x) = \int_0^x du\, f(u)$ has one positive zero $x = a$, $F(x) < 0$ for $0 < x < a$, $F(x) > 0$ and $F'(x) \geq 0$ for $x > a$, and $F(x) \to \infty$ as $x \to \infty$. Then, there is a <u>unique, stable limit cycle</u> about $\mathbf{0}$ in the phase plane.

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# Extremely Nonlinear Systems

Nonlinear systems can rarely be solved.

Given that a closed orbit exists, what can we say about its shape and period?

Nonlinear ODEs → Find suitable transformation → Study phase portrait

# Van der Pol Oscillator With $\mu \gg 1$

**Ex.** Show that VdP has a unique stable limit cycle, for any positive $\mu$.

**E.g.** (Large damping)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu \gg 1.$$

Let $F(x) = x^3/3 - x$, $y = \dot{x}/\mu + F(x)$.

We study the phase plane $(x, y)$. CHECK:

$$\begin{cases} \dot{x} = \mu(y - F(x)), \\ \dot{y} = - x/\mu. \end{cases}$$

# Two-Time-Scale Problems

Cubic nullcline: $y = F(x)$

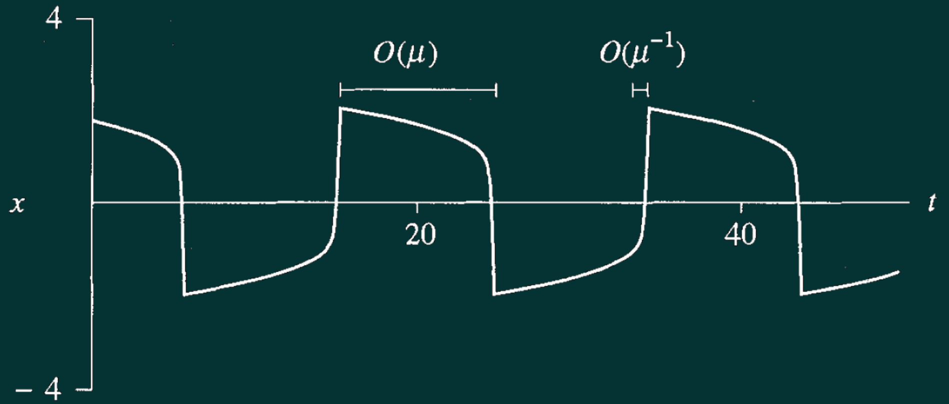
- $y - F(x) = O(1)$: fast

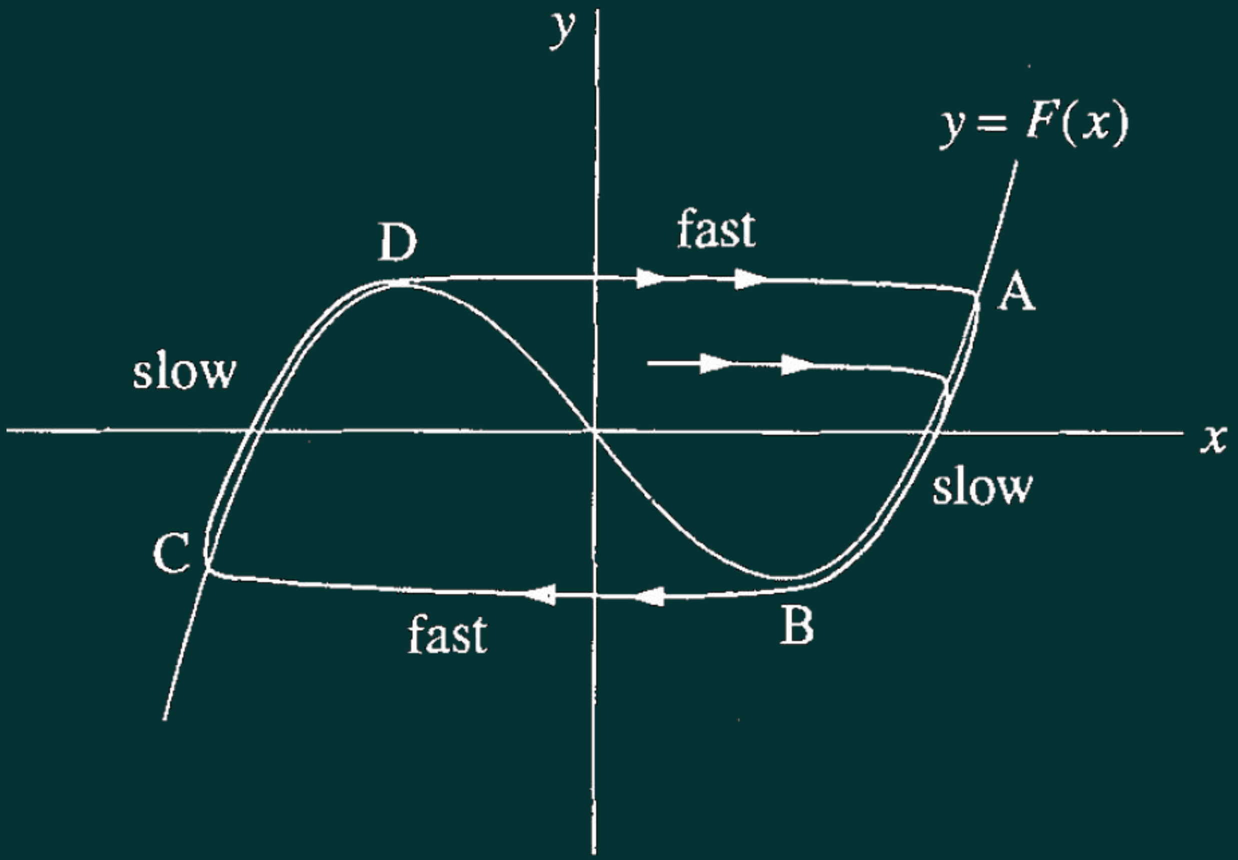
  $$|\dot{x}| = O(\mu) \gg 1,$$

  $$|\dot{y}| = O(\mu^{-1}).$$

- $y - F(x) = O(\mu^{-2})$: slow

  $$|\dot{x}|, |\dot{y}| = O(\mu^{-1})$$




*Strogatz* Fig. 7.5.1 / 7.5.2

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# Estimating Period of VdP

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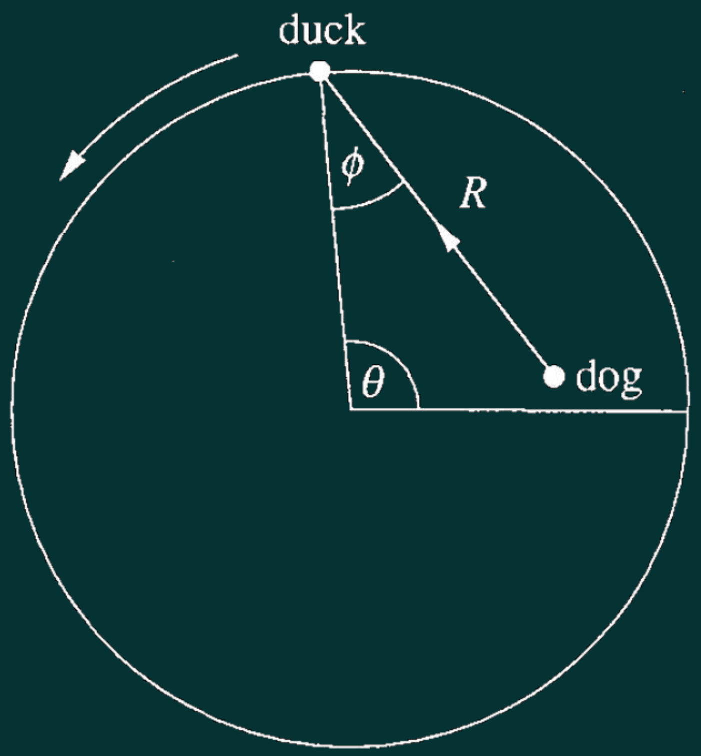
# Exercises -- Pursuit Problem

**Ex.** (*Strogatz* 7.1.9, **circular pursuit problem**)
A dog at the center of circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight toward it. Meanwhile, the duck escapes by swimming around the circumference counterclockwise as fast as it can.

a) Assuming the pond has radius 1 and both animals swim at the same constant speed, derive a pair of ODEs for the path of the dog. (*Hint:* Use the duck coordinate system shown in Figure 2 and find equations for $\mathrm{d}R/\mathrm{d}\theta$ and $\mathrm{d}\varphi/\mathrm{d}\theta$.) Analyze the system. Can you solve it explicitly? Does the dog ever catch the duck?

b) Suppose the dog swims $k$ times faster than the duck. Derive the ODEs for the dog's path.

c) If $k = 3$, what does the dog end up doing in the long run?



# Exercises -- Inter-Species Competition

**Ex.** (*Strogatz* 7.2.13, **competition model**)
Recall the competition model

$$\dot{N} = qN(1 - N/K) - aNM,$$
$$\dot{M} = rM(1 - M/L) - bNM.$$

Here $N$, $M$ are the respective numbers of species 1 and 2. Using Dulac's criterion with the weighting function $g(N, M) = 1/(NM)$, show that the system has no periodic orbits for $N, M > 0$.

This is very similar to a class of models called predator-prey models. Keywords: Lotka-Volterra equation

# What We Left Out

- Duffing equation
- Weakly nonlinear oscillators
- Two-timing method and perturbation
- Fast-slow [supplementary]
- Biased VdP ⟷ Fitzhugh-Nagumo ⟷ Hodgkin-Huxley (neuroscience)
- Auto-parametric amplification
- BIFURCATION!!!
- Symmetry breaking