2025 Fall Introduction to Geometry

Homework 1 (Due Sep 12, 2025)

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Problem 1. 1.2.5 Let $\alpha\colon I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Solution 1. Suppose $|\alpha(t)| = c \neq 0$ for all $t \in I$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\alpha(t)|^2 = 2\alpha(t)\cdot\alpha'(t) = \frac{\mathrm{d}}{\mathrm{d}t}c^2 = 0.$$

Thus $\alpha(t)\cdot\alpha'(t) = 0$, and $\alpha(t)$ and $\alpha'(t)$ are orthogonal. Conversely, suppose $\alpha(t)$ and $\alpha'(t)$ are orthogonal for all $t \in I$, so $\alpha(t)\cdot\alpha'(t) = 0$. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\alpha(t)|^2 = 2\alpha(t)\cdot\alpha'(t) = 0.$$

Thus $|\alpha(t)|$ is a constant.

Problem 2. 1.3.2 A circular disk of radius 1 in the plane xy rolls without slipping along the x-axis. The figure described by a point on the circumference of the disk is called a **cycloid** (Figure 1-7).

a. Obtain a parametrized curve $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.
b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution 2.

a. Let $\alpha(t) = (x(t), y(t))$ be the parametrized curve of the cycloid. As the disk rolls without slipping and the radius of the disk is 1, the distance traveled along the x-axis is t and the y-coordinate is given by the height of the point on the circumference. Therefore, we have:

$$\begin{aligned} x(t) &= t - \sin(t), \\ y(t) &= 1 - \cos(t). \end{aligned} \tag{1}$$

The singular points occur when $\alpha'(t) = 0$. This is equivalent to

$$\begin{aligned} x'(t) &= 1 - \cos(t) = 0, \\ y'(t) &= \sin(t) = 0. \end{aligned} \tag{2}$$

Hence the singular points are at $t = 2n\pi$ for all $n \in \mathbb{Z}$.

b. The arc length of the cycloid for a complete rotation is given by integrating over $[0, 2\pi]$.

$$\begin{aligned} L &= \int_0^{2\pi} \mathrm{d}t\,|\alpha'(t)| = \int_0^{2\pi} \mathrm{d}t\,\sqrt{(1-\cos(t))^2 + (\sin(t))^2} \\ &= \int_0^{2\pi} \mathrm{d}t\,\sqrt{2 - 2\cos(t)} = 8. \end{aligned} \tag{3}$$

Problem 3. 1.3.4 Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t,\, \cos t + \log \tan \frac{t}{2}\right),$$

where t is the angle that the y-axis makes with the vector $\alpha(t)$. The trace of α is called the *tractrix* (see Fig. 1-9). Show that:

a. α is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1.

Solution 3. Recall that a **regular** curve is a smooth, parametrized curve with a non-vanishing derivative.

a. First we shall compute the derivative of $\alpha(t)$ as

$$\begin{aligned} \alpha'(t) &= \left(\cos t,\, -\sin t + \frac{1}{\sin t}\right) \\ &= (\cos t, \cot t \cos t) \end{aligned} \tag{4}$$

Since $\alpha'(t)$ is continuous on $(0, \pi)$ and $\alpha'(t) \neq 0$ for all $t \in (0, \pi)\backslash\{\pi/2\}$, $\alpha(t)$ is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The equation of the tangent line at $\alpha(t)$ is given by

$$y - y_0(t) = \cot t\,(x - x_0(t))\,, \tag{5}$$

where $y_0 = \cos t + \log \tan \frac{t}{2}$ and $x_0 = \sin t$. Setting $x = 0$ to find the intersection with the y-axis, we have

$$\begin{aligned} \Delta y &\equiv y - y_0(t) = -\cot t \sin t = -\cos t, \\ \Delta x &\equiv x - x_0(t) = -\sin t. \end{aligned} \tag{6}$$

Then the distance is $\sqrt{(\Delta y)^2 + (\Delta x)^2} = 1$.

Problem 4. 1.3.8 Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is define as

$$|P| = \max(t_i - t_{i-1}),\ i = 1, \ldots, n$$

Geometrically, $l(\alpha, P)$ is the length of the polygon inscribed in $\alpha([a, b])$ with the vertices in $\alpha(t_i)$. The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of the length of the inscribed polygons. Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| \mathrm{d}t - l(\alpha, P) \right| < \epsilon$$

Solution 4. Since $\alpha(t)$ is differentiable on the closed interval $[a, b]$, $\alpha'(t)$ is continuous. Thus, for any $\epsilon' > 0$ there exists $\delta' > 0$ such that $\alpha'(t_2) - \alpha'(t_1) < \epsilon$ whenever $|t_2 - t_1| < \delta'$. For a partition

P, let $\epsilon' > 0$. The integral can be bounded as:

$$\begin{aligned}
\left| \int_a^b \mathrm{d}t\, |\alpha'(t)| - l(\alpha, P) \right| &= \left| \int_a^b \mathrm{d}t\, |\alpha'(t)| - \sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| \right| \\
&\leq \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \mathrm{d}t |\alpha'(t)| - |\alpha(t_i) - \alpha(t_{i-1})| \right| \\
&\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathrm{d}t\, \left| |\alpha'(t)| - \frac{|\alpha(t_i) - \alpha(t_{i-1})|}{t_i - t_{i-1}} \right| \\
&\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathrm{d}t\, |\alpha'(t) - \alpha'(\xi)| \\
&< n(b-a)\epsilon'.
\end{aligned} \tag{7}$$

whenever $t - \xi < |P| < \min_{i \in \{1,\dots,n\}} (\delta_i')$. We have used the Mean Value Theorem to obtain ξ. Now, let $\epsilon' = \epsilon / n(b-a)$, $\delta = \delta'$, then for any partition P with $|P| < \delta$, we have

$$\left| \int_a^b \mathrm{d}t\, |\alpha'(t)| - l(\alpha, P) \right| < \epsilon. \tag{8}$$

Problem 5 (Exercise 1.4.11)**.** a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an oriented volume in $\mathbb{R}^3$.

b. Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} \tag{1}$$

Solution 5.

a. By definition, the volume of the parallelepiped is given by the area of the base times the height. The area of the base formed by u and v is given by $|u \wedge v|$, and the height is given by the projection of w onto the normal vector of the base, which is $\frac{(u \wedge v)}{|u \wedge v|}$. Therefore, the volume V is given by

$$V = |u \wedge v| \cdot \left| w \cdot \frac{(u \wedge v)}{|u \wedge v|} \right| = |(u \wedge v) \cdot w|. \tag{9}$$

The oriented volume can be introduced as $V = (u \wedge v) \cdot w$. If the vectors u, v, w (in order) form a right-handed system, the oriented volume is positive; otherwise, it is negative.

b. Recall that the vector product $u \wedge v \in \mathbb{R}$ is the unique vector where $(u \wedge v) \cdot w = \det(u, v, w)$. By (a), the volume of the parallelepiped is given by

$$V = |(u \wedge v) \cdot w|. \tag{10}$$

Then,

$$\begin{aligned}
V^2 &= ((u \wedge v) \cdot w)((u \wedge v) \cdot w) \\
&= \det(u, v, w)^2 \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \left| \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \right| \\
&= \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.
\end{aligned} \tag{11}$$