2025 Fall Introduction to Geometry

Homework 10 (Due Nov 28, 2025)

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Definition 1 (isometry). A diffeomorphism $\varphi : S \to \overline{S}$ is an <u>isometry</u> if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have

$$\langle w_1, w_2 \rangle_p = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle_{\varphi(p)}.$$

The surfaces S and $\overline{S}$ are then said to be <u>isometric</u>.

Remark. An isometry is a diffeomorphism that preserves the first fundamental form.

Proposition 1 (Do Carmo Proposition 4.2.1). Assume the existence of parametrizations $\mathbf{x} : U \to S$ and $\overline{\mathbf{x}} : U \to \overline{S}$ such that $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$ in U. Then $\overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{S}$ is a local isometry.

Exercise 1 (Do Carmo 4.2.5). Let $\alpha_1 : I \to \mathbb{R}^3$, $\alpha_2 : I \to \mathbb{R}^3$ be regular parametrized curves, where the parameter is the arc length. Assume that the curvatures k_1 of α_1 and k_2 of α_2 satisfy

$$k_1(s) = k_2(s) \neq 0, \quad s \in I.$$

Let

$$\mathbf{x}_1(s, v) = \alpha_1(s) + v\alpha_1'(s), \qquad \mathbf{x}_2(s, v) = \alpha_2(s) + v\alpha_2'(s)$$

be their (regular) tangent surfaces (cf. Example 5, Sec. 2-3) and let V be a neighborhood of (s_0, v_0) such that $\mathbf{x}_1(V) \subset \mathbb{R}^3$, $\mathbf{x}_2(V) \subset \mathbb{R}^3$ are regular surfaces (cf. Prop. 2, Sec. 2-3). Prove that

$$\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \longrightarrow \mathbf{x}_1(V)$$

is an isometry.

Solution 1. To show that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is an isometry, we need to show that it is a diffeomorphism and preserves the first fundamental form. From Example 2.3.5, the tangent surface of a regular curve α is a regular surface, since for all $(t, v) \subseteq U = \{(t, v) \in I \times \mathbb{R} \mid v \neq 0\}$, we have

$$k(s) = \frac{|\alpha'(s) \wedge \alpha''(s)|}{|\alpha'(s)|^3} \neq 0 \implies \frac{\partial \mathbf{x}}{\partial s} \wedge \frac{\partial \mathbf{x}}{\partial v} = v\alpha''(s) \wedge \alpha'(s) \neq 0.$$

Thus, both $\mathbf{x}_1$ and $\mathbf{x}_2$ are regular parametrizations, and hence homeomorphisms on a small neighborhood $V \subseteq \mathbb{R}^3$. Since $\mathbf{x}$ is differentiable and $\mathrm{d}\mathbf{x}_i$ has full rank, $\mathbf{x}_i^{-1}$ is differentiable for $i = 1, 2$ by the Inverse Function Theorem. Therefore, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is a diffeomorphism. In the Frenet frames of α_i, $i = 1, 2$, we have $\mathbf{x}_i(s, v) = \alpha_i(s) + v\alpha'(s)$, and

$$\mathbf{x}_{i,s} = \alpha'(s) + v\alpha''(s) = T_i(s) + vk_i(s)N_i(s), \quad \mathbf{x}_{i,v} = \alpha'(s) = T_i(s).$$

The first fundamental form coefficients are computed to be

$$E_i = \langle \mathbf{x}_{i,s}, \mathbf{x}_{i,s} \rangle = 1 + v^2 k_i^2(s), \quad F_i = \langle \mathbf{x}_{i,s}, \mathbf{x}_{i,v} \rangle = 1, \quad G_i = \langle \mathbf{x}_{i,v}, \mathbf{x}_{i,v} \rangle = 1.$$

Since $k_1(s) = k_2(s)$ for all $s \in I$, we have $E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$. By Proposition 4.2.1, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is a local isometry. Since $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is also a diffeomorphism, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is an isometry.

Exercise 2 (Do Carmo 4.2.6*). Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with $k(t) \neq 0$, $t \in I$. Let $\mathbf{x}(t, v)$ be its tangent surface. Prove that, for each $(t_0, v_0) \in I \times (\mathbb{R} - \{0\})$, there exists a neighborhood V of (t_0, v_0) such that $\mathbf{x}(V)$ is isometric to an open set of the plane (thus, tangent surfaces are locally isometric to planes).

Solution 2. We will construct the desired local isometry. Fir reparametrize by arc length to get $\alpha(s)$, and define $\mathbf{x}(s, v) = \alpha(s) + v\alpha'(s)$. Let $k(s)$ be the curvature of $\alpha(s)$. As in a previous exercise, let

$$\theta(s) = \int_{s_0}^{s} \mathrm{d}u \, k(u), \quad s_0 \in I$$

be the angle function, and define a plane curve $\beta(s)$ by

$$\beta(s) = \left(\int_{s_0}^{s} \mathrm{d}u \, \cos\theta(u), \int_{s_0}^{s} \mathrm{d}u \, \sin\theta(u), 0 \right),$$

$$\beta'(s) = (\cos\theta(s), \sin\theta(s), 0) \implies |\beta'(s)| = 1,$$

$$\beta''(s) = \theta'(s) \left(-\sin\theta(s), \cos\theta(s), 0\right) = k(s) \left(-\sin\theta(s), \cos\theta(s), 0\right).$$

Then, the curvature of $\beta(s)$ is exactly $k(s)$, and hence $\beta(s)$ is a unit-speed curve with the same curvature as α. Since both β and β' lie in the plane $z = 0$, the image of the tangent surface $\overline{\mathbf{x}}(s, v) = \beta(s) + v\beta'(s)$ is an open subset of the xy-plane. For $\mathbf{x}$ and $\overline{\mathbf{x}}$, we have

$$\mathbf{x}_s = T(s) + vk(s)N(s), \quad \mathbf{x}_v = T(s),$$

$$\overline{\mathbf{x}}_s = \overline{T}(s) + vk(s)\overline{N}(s), \quad \overline{\mathbf{x}}_v = \overline{T}(s),$$

where $T, N, \overline{T}, \overline{N}$ are the tangent vector and normal vector of $\mathbf{x}$ and $\overline{\mathbf{x}}$, respectively. The first fundamental form coefficients of $\mathbf{x}$ and $\overline{\mathbf{x}}$ are, respectively,

$$E = 1 + v^2k^2(s), \quad F = 1, \quad G = 1,$$

$$\overline{E} = 1 + v^2k^2(s), \quad \overline{F} = 1, \quad \overline{G} = 1.$$

Since the coefficients agree, by Proposition 4.2.1, the map $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is a local isometry from $\mathbf{x}(V)$ to an open set of the plane for some neighborhood V of (s_0, v_0). Therefore, the tangent surface is locally isometric to an open set of the plane.

Exercise 3 (Do Carmo 4.2.7). Let V and W be n-dimensional vector spaces with inner products denoted by $\langle \, , \, \rangle$ and let $F : V \to W$ be a linear map. Prove that the following conditions are equivalent:

- **a.** $\langle F(v_1), F(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$.
- **b.** $|F(v)| = |v|$ for all $v \in V$.
- **c.** If $\{v_1, \ldots, v_n\}$ is an orthonormal basis in V, then $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.
- **d.** There exists an orthonormal basis $\{v_1, \ldots, v_n\}$ in V such that $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.

If any of these conditions is satisfied, F is called a linear isometry of V into W. (When $W = V$, a linear isometry is often called an orthogonal transformation.)

Solution 3.

- **a. $\implies$ b.** Suppose $\langle F(v_1), F(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$. Then for all $v \in V$,

$$|v| = \sqrt{\langle v, v \rangle} = \sqrt{\langle F(v), F(v) \rangle} = |F(v)|.$$

- **b. $\implies$ c.** Suppose $|F(v)| = |v|$ for all $v \in V$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Then, for all $i, j = 1, \ldots, n$, since the inner product is induced by a norm $|\cdot|$, we have

$$\begin{aligned}\langle F(v_i), F(v_j)\rangle &= \frac{1}{2}\left(|F(v_i) + F(v_j)|^2 - |F(v_i)|^2 - |F(v_j)|^2\right)\\ &= \frac{1}{2}\left(|v_i + v_j|^2 - |v_i|^2 - |v_j|^2\right) = \langle v_i, v_j\rangle = \delta_{ij}.\end{aligned}$$

  Thus, $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal set in W. Since F is linear, $\{F(v_1), \ldots, F(v_n)\}$ spans $\mathrm{Im}(F)$. Since $\dim(\mathrm{Im}(F)) \leq n$, we have $\dim(\mathrm{Im}(F)) = n$, and hence $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis of W.

- **c. $\implies$ d.** Since V is finite-dimensional, just pick any orthonormal basis of V.

- **d. $\implies$ a.** Suppose there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of V such that $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis of W. For all $v_1, v_2 \in V$, we can write

$$v_1 = \sum_{i=1}^{n} a_i v_i, \quad v_2 = \sum_{j=1}^{n} b_j v_j,$$

  where $a_i, b_j \in \mathbb{R}$. Then,

$$\begin{aligned}\langle F(v_1), F(v_2)\rangle &= \left\langle F\left(\sum_{i=1}^{n} a_i v_i\right), F\left(\sum_{j=1}^{n} b_j v_j\right)\right\rangle\\ &= \left\langle \sum_{i=1}^{n} a_i F(v_i), \sum_{j=1}^{n} b_j F(v_j)\right\rangle\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \langle F(v_i), F(v_j)\rangle\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \delta_{ij} = \sum_{i=1}^{n} a_i b_i = \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j\right\rangle = \langle v_1, v_2\rangle.\end{aligned}$$

Exercise 4 (Do Carmo 4.2.8*)**.** Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that

$$|G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in \mathbb{R}^3$$

(that is, G is a distance-preserving map). Prove that there exists $p_0 \in \mathbb{R}^3$ and a linear isometry (cf. Exercise 7) F of the vector space $\mathbb{R}^3$ such that

$$G(p) = F(p) + p_0 \quad \text{for all } p \in \mathbb{R}^3.$$

Solution 4. Let $p_0 = G(0)$, and let $F(p) = G(p) - p_0$. Then, for all $p, q \in \mathbb{R}^3$, we have

$$|F(p) - F(q)| = |G(p) - G(q)| = |p - q|, \quad F(0) = G(0) - p_0 = 0.$$

Hence F is a distance-preserving map that fixes the origin. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, and $v_i = F(e_i)$ for $i = 1, 2, 3$. Since F is distance-preserving, we have

$$|v_i|^2 = |F(e_i) - F(0)|^2 = |e_i - 0|^2 = 1, \quad |v_i - v_j|^2 = |F(e_i) - F(e_j)|^2 = |e_i - e_j|^2 = 2,$$

squaring both sides gives

$$\langle v_i, v_j\rangle = 0 \text{ for } i \neq j \implies \{v_1, v_2, v_3\} \text{ is an orthonormal basis for } \mathbb{R}^3.$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(e_i) = v_i$ for $i = 1, 2, 3$. Then L is linear by construction, and $L(e_i) = v_i = F(e_i)$, $i = 1, 2, 3$. For any $p \in \mathbb{R}^3$, since $L(0) = 0$, by the distance-preserving property of F, we have $|F(p)| = |p| = |L(p)|$. Then, for all $i = 1, 2, 3$, we have

$$|F(p) - F(e_i)| = |p - e_i| = |L(p) - L(e_i)| .$$

Squaring both sides, then using $|F(p)| = |L(p)|$ and $F(e_i) = L(e_i)$, we have $\langle F(p) - L(p), F(e_i)\rangle = 0$. Hence, $F = L$, and F is linear. By Exercise 4.3.7, F is a linear isometry. Therefore, there exists a linear isometry F such that $G(p) = F(p) + p_0$ for all $p \in \mathbb{R}^3$.

Exercise 5 (Do Carmo 4.2.9)**.** Let S_1, S_2, and S_3 be regular surfaces. Prove that

- **a.** If $\varphi : S_1 \to S_2$ is an isometry, then $\varphi^{-1} : S_2 \to S_1$ is also an isometry.
- **b.** If $\varphi : S_1 \to S_2$, $\psi : S_2 \to S_3$ are isometries, then $\psi \circ \varphi : S_1 \to S_3$ is an isometry.

This implies that the isometries of a regular surface S constitute in a natural way a group, called the group of isometries of S.

Solution 5.

- **a.** Since φ is an isometry, for all $p \in S_1$ and all pairs $w_1, w_2 \in T_p(S_1)$ we have

  $$\langle w_1, w_2\rangle_p = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2)\rangle_{\varphi(p)}.$$

  Let $q = \varphi(p) \in S_2$ and $u_1, u_2 \in T_q(S_2)$. Since φ is a diffeomorphism, $\mathrm{d}\varphi$ is injective. Since the differential $\mathrm{d}\varphi$ is a linear transformation between finite-dimensional spaces, it is also surjective. Thus, there exist $w_1, w_2 \in T_p(S_1)$ such that $\mathrm{d}\varphi_p(w_i) = u_i$ for $i = 1, 2$. Thus,

  $$\langle \mathrm{d}\varphi_p^{-1}(u_1), \mathrm{d}\varphi_p^{-1}(u_2)\rangle_q = \langle w_1, w_2\rangle_p = \langle u_1, u_2\rangle_{\varphi(p)}.$$

  Therefore, φ^{-1} is an isometry.
- **b.** Suppose $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are isometries. Since diffeomorphism between regular surfaces is an equivalence relation (by previous exercise), $\psi \circ \varphi$ is a diffeomorphism. For all $p \in S_1$ and all pairs $w_1, w_2 \in T_p(S_1)$, we have

  $$\begin{aligned}\langle w_1, w_2\rangle_p &= \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2)\rangle_{\varphi(p)} \\ &= \langle \mathrm{d}\psi_{\varphi(p)}(\mathrm{d}\varphi_p(w_1)), \mathrm{d}\psi_{\varphi(p)}(\mathrm{d}\varphi_p(w_2))\rangle_{\psi(\varphi(p))} \\ &= \langle \mathrm{d}(\psi \circ \varphi)_p(w_1), \mathrm{d}(\psi \circ \varphi)_p(w_2)\rangle_{(\psi\circ\varphi)(p)},\end{aligned}$$

  where the chain rule is used in the last equality. Therefore, $\psi \circ \varphi$ is an isometry.

Remark. Since function composition is associative and the identity map $\mathrm{id} : S_1 \to S_1$ is an isometry, by **a.** and **b.**, the set of isometries on S forms a group.