2025 Fall Introduction to Geometry

Homework 11 (Due Dec 5, 2025)

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Theorem 1 (Theorema Egregium)**.** The Gaussian curvature K of a surface is invariant under local isometries. Explicitly, for a parametrization $\mathbf{x}(u, v)$, we have

$$-EK = \left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2.$$

Lemma 1 (Gaussian curvature)**.** The Gaussian curvature K of a regular surface is given by

$$K = \frac{eg - f^2}{EG - F^2}.$$

Proof. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a regular surface S. Then, we have

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ e &= \langle \mathbf{x}_{uu}, N \rangle, \quad f = \langle \mathbf{x}_{uv}, N \rangle, \quad g = \langle \mathbf{x}_{vv}, N \rangle, \end{aligned}$$

where N is the unit normal. In the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, the first and second fundamental forms are

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The shape operator $S : T_pS \to T_pS$ is defined by $S(v) = -\mathrm{d}N_v$, with the principal curvatures k_1, k_2 being its eigenvalues. It has been shown that $S = g^{-1}A$, so

$$K = \det S = \det\left(g^{-1}A\right) = \frac{\det A}{\det g} = \frac{eg - f^2}{EG - F^2}.$$

□

Exercise 4.3.1. Show that if $\mathbf{x}$ is an orthogonal parametrization, that is, $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.$$

Solution 4.3.1. From the definition of the Christoffel symbols, we have

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L_1 N,\\
\mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + L_2 N,\\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + L_3 N,
\end{aligned}$$

we can compute the relations satisfied by the Christoffel symbols by taking inner product with $\mathbf{x}_u$ and $\mathbf{x}_v$ for each of the three equations above. Then, we get

$$\begin{aligned}
&\Gamma^1_{11}E + \Gamma^2_{11}F = \frac{E_u}{2}, \quad \Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \frac{E_v}{2},\\
&\Gamma^1_{12}E + \Gamma^2_{12}F = \frac{E_v}{2}, \quad \Gamma^1_{12}F + \Gamma^2_{12}G = \frac{G_u}{2},\\
&\Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \frac{G_u}{2}, \quad \Gamma^1_{22}F + \Gamma^2_{22}G = \frac{G_v}{2}.
\end{aligned}$$

Since $F = 0$ and $\Gamma^i_{jk} = \Gamma^i_{kj}$, we have

$$\begin{aligned}
&\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E},\\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2E}, \quad \Gamma^2_{22} = \frac{G_v}{2G}.
\end{aligned}$$

and taking inner product with N gives $L_1 = e$, $L_2 = f$, $L_3 = g$. Thus, we have

$$\begin{aligned}
\mathbf{x}_{uu} &= \frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN,\\
\mathbf{x}_{uv} &= \frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN,\\
\mathbf{x}_{vv} &= -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + gN.
\end{aligned}$$

Next, use equation (1) in Section 4.3 to get

$$\begin{aligned}
N_u &= \frac{fF - eG}{EG - F^2}\mathbf{x}_u + \frac{eF - fE}{EG - F^2}\mathbf{x}_v = -\frac{e}{E}\mathbf{x}_u - \frac{f}{G}\mathbf{x}_v,\\
N_v &= \frac{gF - fG}{EG - F^2}\mathbf{x}_u + \frac{fF - gE}{EG - F^2}\mathbf{x}_v = -\frac{f}{E}\mathbf{x}_u - \frac{g}{G}\mathbf{x}_v.
\end{aligned}$$

Since the parametrization is continuously differentiable, the partial derivatives commute, and we have $\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = 0$. First, let's compute the following partial derivatives:

$$\left(\frac{E_v}{2G}\right)_v = \frac{E_{vv}}{2G} - \frac{E_v G_v}{2G^2}, \quad \left(\frac{G_u}{2G}\right)_u = \frac{G_{uu}}{2G} - \frac{(G_u)^2}{2G^2}.$$

Next, we will compute $\mathbf{x}_{uuv}$:

$$\begin{aligned}
\mathbf{x}_{uuv} = (x_{uu})_v &= \left(\frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN\right)_v \\
&= \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\mathbf{x}_{uv} - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v - \frac{E_v}{2G}\mathbf{x}_{vv} + e_v N + eN_v \\
&= \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\left[\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right] - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v \\
&\quad - \frac{E_v}{2G}\left[-\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + gN\right] + e_v N + e\left(-\frac{f}{E}\mathbf{x}_u - \frac{g}{E}\mathbf{x}_v\right) \\
&= \left[\left(\frac{E_u}{2E}\right)_v + \frac{E_u E_v}{4E^2} + \frac{E_v G_u}{4EG} - \frac{ef}{E}\right]\mathbf{x}_u + \left[-\left(\frac{E_v}{2G}\right)_v + \frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} - \frac{eg}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{E_u f}{2E} - \frac{E_v g}{2G} + e_v\right] N.
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
\mathbf{x}_{uvu} = (x_{uv})_u &= \left(\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right)_u \\
&= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\mathbf{x}_{uu} + \left(\frac{G_u}{2G}\right)_u \mathbf{x}_v + \frac{G_u}{2G}\mathbf{x}_{uv} + f_u N + fN_u \\
&= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\left[\frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN\right] \\
&\quad + \left(\frac{G_u}{2G}\right)_u \mathbf{x}_v + \frac{G_u}{2G}\left[\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right] + f_u N + f\left(-\frac{e}{G}\mathbf{x}_u - \frac{f}{G}\mathbf{x}_v\right) \\
&= \left[\left(\frac{E_v}{2E}\right)_u + \frac{E_u E_v}{4E^2} + \frac{E_v G_u}{4EG} - \frac{ef}{E}\right]\mathbf{x}_u + \left[\left(\frac{G_u}{2G}\right)_u - \frac{(E_v)^2}{4EG} + \frac{(G_u)^2}{4G^2} - \frac{f^2}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{E_v e}{2E} + \frac{G_u f}{2G} + f_u\right] N.
\end{aligned}$$

Combining the two results above, we have

$$\begin{aligned}
\mathbf{x}_{uuv} - \mathbf{x}_{uvu} &= \left[\left(\frac{E_u}{2E}\right)_v - \left(\frac{E_v}{2E}\right)_u\right]\mathbf{x}_u + \left[\frac{E_u f - E_v e}{2E} - \frac{E_v g - G_u f}{2G} + e_v - f_u\right] N \\
&\quad + \left[\frac{E_u G_u + (E_v)^2}{4EG} - \frac{E_v G_v + (G_u)^2}{4G^2} - \frac{eg - f^2}{G} - \left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u\right]\mathbf{x}_v = 0.
\end{aligned}$$

Since $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ is an orthonormal basis, each coefficient is equal to zero. Set the coefficient of $\mathbf{x}_v$ to zero and recall the formula for the Gaussian curvature:

$$\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} = \frac{eg - f^2}{EG} \\
&= \frac{E_u G_u + (E_v)^2}{4E^2 G} - \frac{E_v G_v + (G_u)^2}{4EG^2} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u \\
&= \frac{E_u G_u}{4E^2 G} + \frac{(E_v)^2}{4E^2 G} - \frac{E_v G_v}{4EG^2} - \frac{(G_u)^2}{4EG^2} - \frac{E_{vv}}{2EG} + \frac{E_v G_v}{2EG^2} - \frac{G_{uu}}{2EG} + \frac{(G_u)^2}{2EG^2} \\
&= -\frac{1}{2\sqrt{EG}}\left[\frac{G_{uu}}{\sqrt{EG}} - \frac{E_u G_u}{2E\sqrt{EG}} - \frac{(G_u)^2}{2G\sqrt{EG}} + \frac{E_{vv}}{\sqrt{EG}} - \frac{(E_v)^2}{2E\sqrt{EG}} - \frac{E_v G_v}{2G\sqrt{EG}}\right] \\
&= -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.
\end{aligned}$$

Remark. The above formual for the Gaussian curvature of orthogonal parametrizations is known as the Brioschi formula.

Exercise 4.3.2. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\,\Delta(\log\lambda),$$

where $\Delta\varphi$ denotes the Laplacian $(\partial^2\varphi/\partial u^2) + (\partial^2\varphi/\partial v^2)$ of the function φ. Conclude that when

$$E = G = (u^2 + v^2 + c)^{-2} \quad \text{and} \quad F = 0,$$

then $K = \text{const.} = 4c$.

Solution 4.3.2. Suppose $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$. Then we have

$$\begin{aligned} E_v &= \lambda_v, \quad G_u = \lambda_u, \\ E_{vv} &= \lambda_{vv}, \quad G_{uu} = \lambda_{uu}. \end{aligned}$$

From the proof of Exercise 4.3.1, since an isothermal parametrization is orthogonal, we have

$$\begin{aligned} K &= -\frac{1}{2\sqrt{EG}}\left[\frac{G_{uu}}{\sqrt{EG}} - \frac{E_u G_u}{2E\sqrt{EG}} - \frac{(G_u)^2}{2G\sqrt{EG}} + \frac{E_{vv}}{\sqrt{EG}} - \frac{(E_v)^2}{2E\sqrt{EG}} - \frac{E_v G_v}{2G\sqrt{EG}}\right] \\ &= -\frac{1}{2\lambda}\left[\frac{\lambda_{uu}}{\lambda} - \frac{\lambda_u^2}{2\lambda^2} - \frac{\lambda_u^2}{2\lambda^2} + \frac{\lambda_{vv}}{\lambda} - \frac{\lambda_v^2}{2\lambda^2} - \frac{\lambda_v^2}{2\lambda^2}\right] \\ &= -\frac{1}{2\lambda}\left[\frac{\lambda_{uu} + \lambda_{vv}}{\lambda} - \frac{\lambda_u^2 + \lambda_v^2}{\lambda^2}\right] = -\frac{1}{2\lambda}\Delta(\log\lambda), \end{aligned}$$

since

$$\Delta(\log\lambda) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)(\log\lambda) = \frac{\partial}{\partial u}\left(\frac{\lambda_u}{\lambda}\right) + \frac{\partial}{\partial v}\left(\frac{\lambda_v}{\lambda}\right) = \frac{\lambda_{uu} + \lambda_{vv}}{\lambda} - \frac{\lambda_u^2 + \lambda_v^2}{\lambda^2}.$$

Let $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then we have $\lambda(u, v) = (u^2 + v^2 + c)^{-2}$. Then,

$$\begin{aligned} \frac{\partial}{\partial u}(\log\lambda) &= -2\frac{\partial}{\partial u}\log\left(u^2 + v^2 + c\right) = -\frac{4u}{u^2 + v^2 + c}, \\ \frac{\partial^2}{\partial u^2}(\log\lambda) &= -4\frac{\partial}{\partial u}\left(\frac{u}{u^2 + v^2 + c}\right) = -4\,\frac{(-u^2 + v^2 + c)}{(u^2 + v^2 + c)^2}, \\ \frac{\partial}{\partial v}(\log\lambda) &= -2\frac{\partial}{\partial v}\log\left(u^2 + v^2 + c\right) = -\frac{4v}{u^2 + v^2 + c}, \\ \frac{\partial^2}{\partial v^2}(\log\lambda) &= -4\frac{\partial}{\partial v}\left(\frac{v}{u^2 + v^2 + c}\right) = -4\,\frac{(u^2 - v^2 + c)}{(u^2 + v^2 + c)^2}. \end{aligned}$$

$$\Longrightarrow K = -\frac{1}{2\lambda}\Delta(\log\lambda) = -\frac{1}{2}(u^2 + v^2 + c)^2\left(-\frac{8c}{(u^2 + v^2 + c)^2}\right) = 4c.$$

This surface has constant Gaussian curvature $K = 4c$.

Remark. For $c > 0$, this correponds to the stereographic projection of a sphere of radius $1/\sqrt{c}$ minus the north pole; for $c = 0$, this corresponds to the Euclidean plane; and for $c < 0$, this corresponds to the stereographic projection of a hyperbolic plane.

Exercise 4.3.3. Verify that the surfaces

$$\mathbf{x}(u, v) = (u\cos v,\ u\sin v,\ \log u), \quad u > 0,$$

$$\bar{\mathbf{x}}(u, v) = (u\cos v,\ u\sin v,\ v),$$

have equal Gaussian curvature at the points $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(u, v)$, but that the mapping $\bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

Solution 4.3.3. First, we compute the first fundamental form of $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$:

$$\begin{aligned}
\mathbf{x}_u &= \left(\cos v, \sin v, \frac{1}{u}\right), \quad \mathbf{x}_v = (-u\sin v, u\cos v, 0)\,, \\
E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle = \cos^2 v + \sin^2 v + \frac{1}{u^2} = 1 + \frac{1}{u^2}, \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle = -u\cos v\sin v + u\sin v\cos v + 0 = 0, \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\overline{\mathbf{x}}_u &= (\cos v,\, \sin v,\, 0), \quad \overline{\mathbf{x}}_v = (-u\sin v,\, u\cos v,\, 1), \\
\overline{E} &= \langle \overline{\mathbf{x}}_u,\, \overline{\mathbf{x}}_u\rangle = \cos^2 v + \sin^2 v + 0 = 1, \\
\overline{F} &= \langle \overline{\mathbf{x}}_u,\, \overline{\mathbf{x}}_v\rangle = -u\cos v\sin v + u\sin v\cos v + 0 = 0, \\
\overline{G} &= \langle \overline{\mathbf{x}}_v,\, \overline{\mathbf{x}}_v\rangle = u^2\sin^2 v + u^2\cos^2 v + 1 = u^2 + 1.
\end{aligned}$$

Notice that for orthogonal parametrizations, the Gaussian curvature only depends on the following quantities:

$$E_v = \overline{E}_v = 0, \quad G_u = \overline{G}_u = 2u, \quad EG = \left(1 + \frac{1}{u^2}\right)u^2 = u^2 + 1 = \overline{E}\,\overline{G}.$$

Since $F = \overline{F} = 0$, both parametrizations are orthogonal, so by Exercise 4.3.1 the Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$ are equal. Consider the map $\Phi : S \to \overline{S}$ defined by $\Phi = \overline{\mathbf{x}} \circ \mathbf{x}^{-1}$, where S and $\overline{S}$ are the images of $\mathbf{x}$ and $\overline{\mathbf{x}}$, respectively. Since Φ satisfies $\Phi(\mathbf{x}(u,v)) = \overline{\mathbf{x}}(u,v)$, we have

$$\mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u) = \frac{\partial}{\partial u}\overline{\mathbf{x}}(u,v) = \overline{\mathbf{x}}_u, \quad \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_v) = \frac{\partial}{\partial v}\overline{\mathbf{x}}(u,v) = \overline{\mathbf{x}}_v.$$

Then, we compute the first fundamental form at $\mathbf{x}(u,v)$ under the map Φ:

$$\langle \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u),\, \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u)\rangle = \langle \overline{\mathbf{x}}_u,\, \overline{\mathbf{x}}_u\rangle = \overline{E} = 1 \neq 1 + \frac{1}{u^2} = E = \langle \mathbf{x}_u,\, \mathbf{x}_u\rangle,$$

so Φ is not an isometry.

Remark. Two regular surfaces with identical Gaussian curvature at corresponding points are not necessarily isometric.

Exercise 4.3.8. Compute the Christoffel symbols for an open set of the plane

a. In Cartesian coordinates.

b. In polar coordinates.

Use the Gauss formula to compute K in both cases.

Solution 4.3.8.

a. An open set of the plane can be parametrized in Cartesian coordinates as $\mathbf{x}(u,v) = (u,v,0)$. Then, we have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = 1, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = 1.$$

Since $F = 0$ and $E, G \neq 0$, we have

$$\begin{aligned}
\Gamma^1_{11} &= \frac{E_u}{2E} = 0, \quad \Gamma^2_{11} = -\frac{E_v}{2G} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E} = 0, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{G_u}{2G} = 0, \quad \Gamma^1_{22} = -\frac{G_u}{2E} = 0, \quad \Gamma^2_{22} = \frac{G_v}{2G} = 0.
\end{aligned}$$

Hence, all Christoffel symbols are zero. Next, compute

$$\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = 0,$$

so with the unit normal $N = (0, 0, 1)$, we have

$$e = \langle \mathbf{x}_{uu}, N\rangle = 0, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = 0.$$

Therefore, since $EG - F^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as

$$K = \frac{eg - f^2}{EG - F^2} = 0.$$

b. An open set of the plane can also be parametrized in polar coordinates, given by the parametrization $\mathbf{x}(u, v) = (u \cos v, u \sin v, 0)$. Then, we have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = 1, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = u^2.$$

Since $F = 0$, we have the following Christoffel symbols whenever $u \neq 0$:

$$\Gamma^1_{11} = \frac{E_u}{2E} = 0, \quad \Gamma^2_{11} = -\frac{E_v}{2G} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E} = 0,$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G} = \frac{1}{u}, \quad \Gamma^1_{22} = -\frac{G_u}{2E} = -u, \quad \Gamma^2_{22} = \frac{G_v}{2G} = 0.$$

Unlike in the Cartesian coordinates, not all Christoffel symbols are zero. Next, compute

$$\mathbf{x}_{uu} = (0, 0, 0), \quad \mathbf{x}_{uv} = (-\sin v, \cos v, 0), \quad \mathbf{x}_{vv} = (-u\cos v, -u \sin v, 0),$$

so with the unit normal $N = (0, 0, 1)$, we have

$$e = \langle \mathbf{x}_{uu}, N\rangle = 0, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = 0.$$

Therefore, since $EG - F^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as

$$K = \frac{eg - f^2}{EG - F^2} = 0.$$

Exercise 4.3.9. Justify why the surfaces below are not pairwise locally isometric:

a. Sphere.

b. Cylinder.

c. Saddle $z = x^2 - y^2$.

Solution 4.3.9.

a. The sphere has constant positive Gaussian curvature. Let a sphere of radius r be centered about the origin, and let $\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$ be a parametrization of the sphere. Then,

$$\mathbf{x}_\theta = (r\cos\theta\cos\phi,\ r\cos\theta\sin\phi,\ -r\sin\theta),$$
$$\mathbf{x}_\phi = (-r\sin\theta\sin\phi,\ r\sin\theta\cos\phi,\ 0),$$

and we have

$$E = r^2, \quad F = 0, \quad G = r^2 \sin^2\theta.$$

We can compute

$$E_\phi = 0, \quad G_\theta = 2r^2 \sin\theta\cos\theta, \quad EG = r^4 \sin^2\theta.$$

Then,

$$\left(\frac{E_\phi}{\sqrt{EG}}\right)_\phi = 0, \quad \left(\frac{G_\theta}{\sqrt{EG}}\right)_\theta = (\cos\theta)_\theta = -\sin\theta.$$

Since $F = 0$, the parametrization is orthogonal. By Exercise 4.3.1 we have

$$K = -\frac{1}{2r^2\sin\theta}(-\sin\theta) = \frac{1}{2r^2} > 0.$$

b. The cylinder has zero Gaussian curvature. Let a cylinder of radius r be centered about the z-axis, and let $\mathbf{x}(\theta, z) = (r\cos\theta, r\sin\theta, z)$ be a parametrization of the cylinder. Then,

$$\mathbf{x}_\theta = (-r\sin\theta,\, r\cos\theta,\, 0), \quad \mathbf{x}_z = (0,\, 0,\, 1),$$

and we have

$$E = r^2, \quad F = 0, \quad G = 1.$$

We can compute

$$E_z = 0, \quad G_\theta = 0, \quad EG = r^2.$$

Then,

$$\left(\frac{E_z}{\sqrt{EG}}\right)_z = 0, \quad \left(\frac{G_\theta}{\sqrt{EG}}\right)_\theta = 0.$$

Since $F = 0$, the parametrization is orthogonal. By Exercise 4.3.1 we have

$$K = -\frac{1}{2r}(0+0) = 0.$$

c. The saddle has negative Gaussian curvature. Let the saddle be given by the parametrization $\mathbf{x}(u, v) = (u, v, u^2 - v^2)$. Then,

$$\mathbf{x}_u = (1, 0, 2u), \quad \mathbf{x}_v = (0, 1, -2v),$$
$$\mathbf{x}_{uu} = (0, 0, 2), \quad \mathbf{x}_{uv} = (0, 0, 0), \quad \mathbf{x}_{vv} = (0, 0, -2),$$

and we have $E = 1 + 4u^2$, $F = -4uv$, and $G = 1 + 4v^2$. The normal vector of the surface is given by

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} = \frac{(-2u, 2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}.$$

Then, we have

$$e = \langle \mathbf{x}_{uu}, N\rangle = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}}, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}}.$$

Since $EG - F^2 = (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\left(\frac{2}{\sqrt{1 + 4u^2 + 4v^2}}\right)\left(\frac{-2}{\sqrt{1 + 4u^2 + 4v^2}}\right) - 0}{1 + 4u^2 + 4v^2} = \frac{-4}{(1 + 4u^2 + 4v^2)^2} < 0.$$

Suppose **a.** to **c.** are pairwise locally isometric, then by the Theorema Egregium they must have identical Gaussian curvature at corresponding points, a contradiction to our above calculation.