2025 Fall Introduction to Geometry

Homework 2 (Due Sep 19, 2025)

物理、數學三 黃紹凱 B12202004

September 19, 2025

Remark (1)**.** In the particular case of a plane curve $\alpha : I \to \mathbb{R}^2$, it is possible to give the curvature k a sign. For that, let $\{e_1, e_2\}$ be the natural basis (see Sec. 1-4) of $\mathbb{R}^2$ and define the normal vector $n(s)$, $s \in I$, by requiring the basis $\{t(s), n(s)\}$ to have the same orientation as the basis $\{e_1, e_2\}$. The curvature k is then defined by

$$\frac{dt}{ds} = kn$$

and might be either positive or negative. It is clear that $|k|$ agrees with the previous definition and that k changes sign when we change either the orientation of α or the orientation of $\mathbb{R}^2$ (Fig. 1-16).

Problem 1 (Do Carmo 1.5.7)**.** Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrized plane curve (arbitrary parameter), and define $n = n(t)$ and $k = k(t)$ as in Remark 1. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)}\, n(t), \qquad t \in I, \tag{1}$$

is called the evolute of α (Fig. 1–17).

a. Show that the tangent at t of the evolute of α is the normal to α at t.

b. Consider the normal lines of α at two neighboring points t_1, t_2, $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.

Solution 1.

a. Let β be the evolute. By the chain rule, we have

$$n'(t) = \frac{\mathrm{d}n}{\mathrm{d}s}\frac{\mathrm{d}s}{t} = -k(t)\frac{\alpha'(t)}{|\alpha'(t)|}|\alpha'(t)| = -k(t)\,\alpha'(t).$$

By direct differentiation of β, we get

$$\beta'(t) = \alpha'(t) + \frac{-k(t)^2\,\alpha'(t) - n(t)\,k(t)}{k(t)^2} = -\frac{k'(t)}{k(t)^2}n(t).$$

Hence, the tangent at t of β is precisely $n(t)$.

b. Let the normal be given by $n(t) = (a(t), b(t))$, then $a'(t) \neq 0$ or $b'(t) \neq 0$ for all t since α is regular. Take some $t_2 \in I$, assume without loss of generality that $a'(t_2) \neq 0$. For $t \in J = (t_2 - \delta, t_2 + \delta)$, we have

$$|a'(t_2)| - \left|\frac{a_{t_2} - a_t}{t_2 - t}\right| \leq \left|a'(t_2)\frac{a_{t_2} - a_t}{t_2 - t}\right| < \frac{1}{2}|a'(t_2)|,$$

and

$$\left|\frac{a(t_2) - a(t)}{t_2 - t}\right| > \frac{|a'(t_2)|}{2} > 0,$$

hence $a(t) \neq a(t_2)$ for any t in a neighborhood of t_2. Therefore, if we fix $t_1 \in J$, $t_1 \neq t_2$, then the normal lines N_1, N_2 of α at t_1, t_2 will have a unique intersection. L_1, L_2 are well-defined given that $n(t) \neq 0$ for all $t \in I$. Let $h \in \mathbb{R}^2$ be the intersection point, then

$$h = \alpha(t_1) + p_1 n(t_1) = \alpha(t_2) + p_2 n(t_2),$$

where $p_1, p_2 \in I$ are constants. We shall show that as $t_1 \to t_2$, $p_2 \to 1/k(t_2)$. The area spanned by $n(t_1)$ and $\alpha(t_1)$ is

$$\det(\alpha(t_1), n(t_1)) = \det(\alpha(t_2), n(t_1)) + p_1 \det(n(t_2), n(t_1)),$$

then

$$p_2 = \frac{\det(\alpha(t_1) - \alpha(t_2),\ n(t_1))}{\det(n(t_2),\ n(t_1))}.$$

Taking the limit $t_1 \to t_2$ gives, by L'Hôpital's rule,

$$\begin{aligned}
\lim_{t_1 \to t_2} p_2 = \frac{\det(\alpha'(t_2), n(t_2))}{\det(n(t_2), n'(t_2))} &= \frac{1}{k(t_2)} \\
&= \lim_{t_1 \to t_2} \frac{\det(\alpha'(t_1),\ n(t_1)) - \det(\alpha(t_1) - \alpha(t_2),\ -k(t_1)\,\alpha'(t_1))}{\det(n(t_2), -k(t_1)\,\alpha'(t_1))} \\
&= \lim_{t_1 \to t_2} \frac{|\alpha'(t_1)|}{k(t_1)\,||} + \lim_{t_1 \to t_2} \frac{\det(k(t_1)\,\alpha'(t_1),\ \alpha(t_1) - \alpha(t_2))}{k(t_1)\,|\alpha'(t_1)|} \\
&= \frac{1}{k(t_2)}.
\end{aligned}$$

Therefore,

$$\lim_{t_1 \to t_2} h = \alpha(t_2) + \frac{1}{k(t_2)} n(t_2) = \beta(t_2),$$

which is a point on the evolute of α.

Problem 2 (Do Carmo 1.5.8)**.** The trace of the parametrized curve (arbitrary parameter)

$$\alpha(t) = (t, \cosh t), \qquad t \in \mathbb{R}, \tag{2}$$

is called the catenary.

a. Show that the signed curvature (cf. Remark 1) of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}. \tag{3}$$

b. Show that the evolute (cf. Exercise 7) of the catenary is

$$\beta(t) = \big(t - \sinh t\, \cosh t,\ 2\cosh t\big). \tag{4}$$

Solution 2.

To keep the notation unambiguous, we will denote the (unit) tangent vector by T. Recall that $n(t) = T'(t)/|T'(t)|$, by remark 1, the signed curvature is given by

$$k(t)\, n(t) = \frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{T'(t)}{|\alpha'(t)|}. \tag{5}$$

Plugging in the expression for $n(t)$ simplifies it to

$$k(t) = \frac{|T'(t)|}{|\alpha'(t)|}. \tag{6}$$

a. We have $\alpha'(t) = (1, \sinh t)$, $|\alpha'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t$. Then $T(t) = \alpha'(t)/|\alpha'(t)| = \operatorname{sech} t(1, \sinh t)$ and

$$T'(t) = \operatorname{sech}^2 t\,(-\sinh t, 1)\,,$$
$$|T'(t)| = \operatorname{sech}^2 t\sqrt{\sinh^2 t + 1} = \operatorname{sech} t,$$

By equation (6), we have

$$k(t) = \frac{\operatorname{sech} t}{\cosh t} = \operatorname{sech}^2 t = \frac{1}{\cosh^2 t}. \tag{7}$$

b. By definition in Exercise 7, the evolute is given by

$$\begin{aligned}\beta(t) &= \alpha(t) + \frac{1}{k(t)} n(t) \\ &= (t, \cosh t) + \cosh^2 t \operatorname{sech} t(-\sinh t, 1) \\ &= (t - \sinh t \cosh t, 2\cosh t).\end{aligned} \tag{8}$$

Problem 3 (Do Carmo 1.5.9)**.** Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \mathrm{d}s\, \cos\theta(s) + a,\ \int \mathrm{d}s\, \sin\theta(s) + b\right), \tag{9}$$

where

$$\theta(s) = \int \mathrm{d}s\, k(s) + \varphi, \tag{10}$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

Solution 3. Let $\alpha(s)$ be as given, we have

$$\alpha'(s) = (\cos\theta(s), \sin\theta(s)) = \left(\cos\left(\int k(s)\,ds + \varphi\right), \sin\left(\int k(s)\,ds + \varphi\right)\right), \tag{11}$$

and

$$\alpha''(s) = k(s)\,(-\sin\theta(s), \cos\theta(s))\,, \tag{12}$$

hence $|\alpha''(s)| = k(s)$. By the definition of translation, the curve is determined up to a translation of the vector (a, b), so suppose $a = b = 0$. Now suppose we rotate the curve by an angle φ counterclockwise, then the new curve $\tilde{\alpha}(s)$ is given by

$$\begin{aligned}\tilde{\alpha}(s) &= \begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}\alpha(s) \\ &= \begin{pmatrix}\cos\varphi \int \mathrm{d}s\, \cos\theta(s) - \sin\varphi \int \mathrm{d}s\, \sin\theta(s) \\ \sin\varphi \int \mathrm{d}s\, \cos\theta(s) + \cos\varphi \int \mathrm{d}s\, \sin\theta(s)\end{pmatrix} \\ &= \begin{pmatrix}\int \mathrm{d}s\, \cos(\theta(s) + \varphi) \\ \int \mathrm{d}s\, \sin(\theta(s) + \varphi)\end{pmatrix}.\end{aligned}$$

Thus, the curve is determined up to an arbitrary rotation of the angle φ.

Remark. This exercises shows how to construct a curve with any given curvature functions $k(s)$, up to a translation and rotation. This is a special case of the **Fundamental Theorem of the Local Theory of Curves**.

Problem 4 (Do Carmo 1.5.11)**.** One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$.

a. Show that the arc length is

$$\int_a^b \mathrm{d}\theta \sqrt{\rho^2 + (\rho')^2}, \tag{13}$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\big((\rho')^2 + \rho^2\big)^{3/2}}. \tag{14}$$

Solution 4.

a. Calculate the curve vector in Cartesian coordinates:

$$\alpha(\theta) = (\rho(\theta)\cos\theta,\ \rho(\theta)\sin\theta),$$

Then

$$\alpha'(\theta) = (\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta,\ \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta),$$

and computing the norm gives

$$|\alpha'(\theta)| = \sqrt{(\rho'(\theta))^2 + \rho^2(\theta)}.$$

The arclength is defined to be

$$s(a,b) = \int_a^b \mathrm{d}\theta\, |\alpha'(\theta)| = \int_a^b \mathrm{d}\theta \sqrt{\rho^2 + (\rho')^2}. \tag{15}$$

b. The unit tangent is

$$T(\theta) = \frac{\alpha'(\theta)}{|\alpha'(\theta)|} = \frac{1}{\sqrt{(\rho')^2 + \rho^2}}(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta,\ \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta).$$

Then we calculate $T'(\theta)$ and its magnitude, where prime denotes derivative with respect to θ. After some cumbersome algebra, we get

$$T'(\theta) = \frac{1}{((\rho')^2 + \rho^2)^{3/2}}\left((2(\rho')^2 - \rho\rho'' + \rho^2)(-\sin\theta,\ \cos\theta)\right),$$

By equation (6), we have

$$k(\theta) = \frac{|T'(\theta)|}{|\alpha'(\theta)|} = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{((\rho')^2 + \rho^2)^{3/2}}. \tag{16}$$

Problem 5 (Do Carmo 1.5.14)**.** Let $\alpha : (a,b) \to \mathbb{R}^2$ be a regular parametrized plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $|\alpha(t)|$ from the origin to the trace of α will be a maximum at t_0. Prove that the curvature k of α at t_0 satisfies

$$|k(t_0)| \ge \frac{1}{|\alpha(t_0)|}.$$

Solution 5. Notice that $f(t) = |\alpha(t)|$ is nonnegative, so $f^2(t) = \alpha(t)\cdot\alpha(t)$ also attains a maximum at t_0. Then

$$\frac{d}{dt} f^2(t)\Big|_{t=t_0} = 2\alpha(t_0)\cdot\alpha'(t_0) = 0,$$

differentiating again gives

$$\frac{d^2}{dt^2} f^2(t)\Big|_{t=t_0} = \alpha'(t_0) \cdot \alpha'(t_0) + \alpha(t_0) \cdot \alpha''(t_0) \leq 0,$$

since $f(t)$ attains a maximum at t_0. We also have $\alpha'(t_0) \cdot \alpha'(t_0) = 1$ since it is a parametrization by arclength, and $\alpha''(t_0) = k(t_0)n(t_0)$. Then let θ be the angle between $\alpha(t_0)$ and α'', we have

$$k(t_0)n(t_0)\alpha(t_0) = |k(t_0)||n(t_0)||\alpha(t_0)| \cos\theta \leq -1.$$

Notice that $|n(t_0)| = 1$ and $\cos\theta < 0$, we have

$$k(t_0) \geq \frac{1}{|\alpha(t_0)\cos\theta|} \geq \frac{1}{|\alpha(t_0)|}.$$