2025 Fall Introduction to Geometry

Homework 3 (Due Sep 26, 2025)

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Problem 1 (Do Carmo 1.5.10). Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0, 0, 0), & t = 0 \end{cases} \tag{1}$$

a. Prove that α is a differentiable curve.

b. Prove that α is regular for all t and that the curvature $k(t) \neq 0$, for $t \neq 0$, $t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c. Show that the limit of the osculating planes as $t \to 0, t > 0$, is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0, t < 0$, is the plane $z = 0$ (this implies that the normal vector is discontinuous at $t = 0$ and shows why we excluded points where $k = 0$).

d. Show that τ can be defined so that $\tau \equiv 0$, even though α is not a plane curve.

Solution 1.

(a) The curve α is differentiable if α' exists everywhere. For $t > 0$ and $t < 0$ it is made of elementary functions, so it is differentiable. At $t = 0$, the x coordinate is differentiable, so consider the z coordinateo only.

Lemma 1. The map

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0; \\ 0, & x \leq 0. \end{cases} \tag{2}$$

is differentiable at $x = 0$ and $f^{(n)}(0) = 0$.

Proof. Let $f(x) = e^{-1/x^2}$, notice that

$$f(x) \leq n! x^{2n} \quad \text{for all } n. \tag{3}$$

Thus, for $n = 1$ we have $f'(0) = \lim_{x\to 0} f(x)/x = 0$ by the squeeze theorem. Assume that $f^{(k)}(0) = 0$ for all $k < n$. By induction we know that $f^{(k)}$ is of the form $f^{(m)}(x) = f(x)\sum_{r=1}^{N} a_r x^{-r}$ for $x > 0$, so choosing some n large enough such that

$$f^{(k+1)}(x) \leq n! x^{2n} \sum_{r=1}^{N} a_r x^{-r} \leq C x^m$$

for some constant C, we have f is $(k+1)$ times differentiable and $f^{(k+1)}(0) = 0$. By induction we are done. □

By Lemma (1), α is differentiable.

(b) The curve has derivative

$$\alpha' = \begin{cases} \left(1, 0, \dfrac{2}{t^3} e^{-1/t^2}\right), & t > 0, \\ \left(1, \dfrac{2}{t^3} e^{-1/t^2}, 0\right), & t < 0, \\ (1, 0, 0), & t = 0. \end{cases}$$

Since e^{-1/t^2} is always positive, $\alpha'(t) \neq 0$ for all t, so α is regular. Next, we compute the curvature $k(t)$.

Lemma 2. For a regular curve $\alpha(t)$, the curvature is given by

$$k(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}. \tag{4}$$

Proof. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular curve. Then, we have $T'(t(s)) = k(t(s))N(t(s))$, where $t(s)$ is the reparametrization by arc length. Then $|T'(t(s))| = k(t(s))|N(t(s))| = k(t(s))$. The left hand side is $\mathrm{d}T/\mathrm{d}s = (\mathrm{d}T/\mathrm{d}t)\,(\mathrm{d}t/\mathrm{d}s) = (\mathrm{d}T/\mathrm{d}t)\,/|\alpha'(t)|$. Moreover,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{|\alpha'|^2\alpha'' - (\alpha' \cdot \alpha'')\,\alpha'}{|\alpha'|^3} = \frac{\alpha' \wedge (\alpha'' \wedge \alpha')}{|\alpha'|^3}. \tag{5}$$

Since $\alpha' \perp \alpha'' \wedge \alpha'$,

$$k(t(s)) = |T'(t(s))| = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}.$$

□

We have $\alpha'(t)$ given above, and

$$\alpha'' = \begin{cases} \left(0, 0, \left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2}\right), & t > 0, \\ \left(0, \left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2}, 0\right), & t < 0, \\ (0,0,0), & t = 0. \end{cases}$$

$$\alpha' \wedge \alpha'' = \begin{cases} \left(0, -\left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2}, 0\right), & t > 0, \\ \left(0, 0, \left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2}\right), & t < 0, \\ (0,0,0), & t = 0. \end{cases}$$

Using Lemma 2, we have

$$k(t) = \begin{cases} \left|\left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2}\right| \Big/ \left(1 + \frac{4}{t^6} e^{-2/t^2}\right)^{3/2}, & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{6}$$

From above we know $k(t) = 0$ when and only when $t = 0$ and $t = \pm\sqrt{2/3}$.

(c) The osculating plane is determined by the normal vector $N(t)$ and the tangent vector $T(t)$. By equation (4) and the definition $\mathrm{d}T(t(s))/\mathrm{d}s = k(t(s))N(t(s))$, the normal vector is

$$\begin{aligned} N(t) &= \frac{1}{k(t)} \frac{\mathrm{d}T(t(s))}{\mathrm{d}s} \\ &= \frac{\alpha'(t) \wedge (\alpha''(t) \wedge \alpha'(t))}{|\alpha'(t)|^4} \cdot \frac{|\alpha'(t)|^3}{|\alpha'(t) \wedge \alpha''(t)|} \\ &= \frac{\alpha'(t) \wedge (\alpha''(t) \wedge \alpha'(t))}{|\alpha'(t)||\alpha'(t) \wedge \alpha''(t)|}. \end{aligned} \tag{7}$$

For $t > 0$, we have

$$N(t) = \left(1 + \frac{4}{t^6} e^{-1/t^2}\right)^{-1/2} \left(-\frac{2}{t^3} e^{-1/t^2}, 0, 1\right)$$

and

$$T(t) = \left(1 + \frac{4}{t^6} e^{-1/t^2}\right)^{-1/2} \left(1, 0, \frac{2}{t^3} e^{-1/t^2}\right),$$

hence $N_P = \lim_{t\to 0^+} T(t)\wedge N(t) = (0,0,1)\wedge(1,0,0) = (0,1,0)$. Furthermore, $\lim_{t\to 0^+} \alpha(t) = (0,0,0)$, so the osculating plane is $y = 0$.

On the other hand, for $t < 0$, we have

$$N(t) = \left(1 + \frac{4}{t^6}e^{-1/t^2}\right)^{-1/2}\left(-\frac{2}{t^3}e^{-1/t^2}, 1, 0\right)$$

and

$$T(t) = \left(1 + \frac{4}{t^6}e^{-1/t^2}\right)^{-1/2}\left(1, \frac{2}{t^3}e^{-1/t^2}, 0\right),$$

hence $N_P = \lim_{t\to 0^-} T(t)\wedge N(t) = (0,1,0)\wedge(1,0,0) = (0,0,-1)$. Furthermore, $\lim_{t\to 0^-} \alpha(t) = (0,0,0)$, so the osculating plane is $z = 0$. Notice that $N(t)$ is discontinuous at $t = 0$, thus undefined there.

(d) Since $k(0) = k(\pm\sqrt{2/3}) = 0$, $N(0)$ and $N(\pm\sqrt{2/3})$ are not well-defined. Therefore, we can define τ to be zero at these points. For $t \neq 0, \pm\sqrt{2/3}$, we have

$$B(t) = T(t) \wedge N(t) = \begin{cases} -(0,1,0), & t > 0, \\ (0,0,1), & t < 0. \end{cases}$$

The binormal vector $B(t)$ is constant on $I\backslash\{0\}$, so $B'(s) = B'(t)\cdot|\alpha'(t)|^{-1} = 0 = \tau(t(s))N(t(s))$. Hence we can choose $\tau(t) \equiv 0$ for $t \in I\backslash\{0, \pm\sqrt{2/3}\}$. This is an example of **a curve with identically zero torsion that is not a plane curve**.

Problem 2 (Do Carmo 1.5.17)**.** In general, a curve α is called a helix if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

*__a.__ α is a helix if and only if $\frac{k}{\tau} = \text{const}$.

*__b.__ α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

*__c.__ α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)\,ds, \frac{a}{c}\int \cos\theta(s)\,ds\right) \tag{8}$$

where $c^2 = a^2 + b^2$, is a helix, and that $\frac{k}{\tau} = \frac{a}{b}$.

Solution 2.

(a) Suppose there exists a vector $v \in \mathbb{R}^3$ such that $v \cdot t(s) = C$ for some constant C. Then

$$\frac{\mathrm{d}t}{\mathrm{d}s} \cdot v = k(s)n(s) \cdot v = 0,$$

so $n(s) \cdot v = 0$. Differentiating again gives

$$\frac{\mathrm{d}n}{\mathrm{d}s} \cdot v = -k(s)t(s) \cdot v + \tau(s)b(s) \cdot v = -k(s)C + \tau(s)b(s) \cdot v = 0.$$

Since $\tau(s) \neq 0$, we have

$$Ck(s)/\tau(s) = (b(s) \cdot v) = (t(s) \wedge n(s)) \cdot v = (v \wedge t(s)) \cdot n(s).$$

Since $t(s), v \perp n(s)$, the triple product is equal to $|n(s)||t(s)||v|\sin(C) = |v|\sin C$. Therefore, $k(s)/\tau(s)$ is a constant. Conversely, if $k(s)/\tau(s) = C'$ for some constant C', then we can take $v = t(s) + C'b(s)$, which is a constant vector since

$$\frac{\mathrm{d}v}{\mathrm{d}s} = k(s)n(s) + C'\left(-\tau(s)n(s)\right) = 0.$$

Then

$$\frac{\mathrm{d}t}{\mathrm{d}s} \cdot v = 0.$$

(b) Suppose $\alpha(s)$ is a helix, then there exists a vector $v \in \mathbb{R}^3$ such that $v \cdot t(s) = C$ for some constant C. Let L be a line containing $n(s)$ and passing through $\alpha(s)$. Then $n(s) \cdot v = 0$ by result in part (a), so $L \perp v$, hence parallel to the plane with normal vector v. Conversely, for any point $s \in I$, suppose the line L containing $n(s)$ and passing $\alpha(s)$ is parallel to the plane P with normal vector $v \in \mathbb{R}^3$. Then $n(s) \cdot v = 0$, and

$$\frac{\mathrm{d}T}{\mathrm{d}s} \cdot v = k(s)n(s) \cdot v = 0.$$

Hence $\mathrm{d}T/\mathrm{d}s = \mathrm{d}(T \cdot v)/\mathrm{d}s = 0$, and $T(s) \cdot v = C'$ for some constant C', and $\alpha(s)$ is a helix.

(c) By definition of helix, there exists a vector $v \in \mathbb{R}^3$ such that $v \cdot t(s) = C$ for some constant C. By (b), all the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to the plane with some fixed normal vector $u \in \mathbb{R}^3$, so $n(s) \cdot u = 0$. Consider $b \cdot (u \wedge v) = (t(s) \wedge n(s)) \cdot (u \wedge v) = (t(s) \cdot u)(n(s) \cdot v) - (t(s) \cdot v)(n(s) \cdot u) = 0$, since $n(s) \cdot v = 0$ from (a). Conversely, suppose there exists a vector $v \in \mathbb{R}^3$ such that $b(s) \cdot v = C$ for some constant C. Then $(t(s) \wedge n(s)) \cdot v = C$,

$$\frac{\mathrm{d}b}{\mathrm{d}s} \cdot v = -\tau(s)n(s) \cdot v = 0,$$

and by $\tau(s) \neq 0$ we have $n(s) \cdot v = 0$. Finally,

$$\frac{\mathrm{d}}{\mathrm{d}s}(t(s) \cdot v) = k(s)n(s) \cdot v = 0,$$

therefore, $\alpha(s)$ is a helix.

(d) With s suppressed in the expressions, derivatives of α are

$$\begin{aligned}
\alpha' &= \left(\frac{a}{c}\sin\theta(s), \frac{a}{c}\cos\theta(s), \frac{b}{c}\right), \\
\alpha'' &= \left(\frac{a}{c}\theta'(s)\cos\theta(s), -\frac{a}{c}\theta'(s)\sin\theta(s), 0\right), \\
\alpha''' &= \left(\frac{a}{c}\left(\theta''(s)\cos\theta(s) - (\theta'(s))^2\sin\theta(s)\right), -\frac{a}{c}\left(\theta''(s)\sin\theta(s) + (\theta'(s))^2\cos\theta(s)\right), 0\right).
\end{aligned}$$

The curvature is $k(s) = |\alpha'(s)| = \frac{a}{c}\theta'$. The torsion is given by the formula

$$\tau(s) = -\frac{(\alpha'(s) \wedge \alpha''(s)) \cdot \alpha'''(s)}{k(s)^2}$$

by [Do Carmo] Exercise 1.5.2. Direct calculation gives

$$(\alpha' \wedge \alpha'') \cdot \alpha''' = \left(\frac{ab}{c^2}\theta'(s)\sin\theta(s), -\frac{ab}{c^2}\theta'(s)\cos\theta(s), -\frac{a^2}{c^2}(\theta'(s))^2\right) = \frac{a^2 b}{c^3}(\theta')^3,$$

so

$$\tau(s) = \frac{b}{c}\theta'(s) = \frac{b}{a}k(s).$$

Problem 3 (Do Carmo 1.6.1)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length with curvature $k(s) \neq 0$, $s \in I$. Let P be a plane satisfying both of the following conditions:

1. P contains the tangent line at s.
2. Given any neighborhood $J \subset I$ of s, there exist points of $\alpha(J)$ in both sides of P.

Prove that P is the osculating plane of α at s.

Solution 3. Let n be the normal vector of plane P, then condition 1 implies that $n_P \perp t(s)$, as $t(s) \in P$. To show the desired reuslt, we will show that $n(s) \perp n_P$. Consider $f(s) = t(s) \cdot n_P = 0$, differentiating both sides gives $f'(s) = t(s) \cdot n'_P = k(s)n(s) \cdot n_P = 0$, so $n(s) \perp n_P$. Thus, the binormal vector $b(s) \parallel n_P$. Furthermore, by condition 2 we can take some interval $J = \left(s - \frac{1}{m}, s + \frac{1}{m}\right) \subseteq I$, then there exists $s_1^{(m)} \in \left(s - \frac{1}{m}, s\right)$ and $s_2^{(m)} \in \left(s, s + \frac{1}{m}\right)$ such that $\alpha(s_1^{(m)})$ and $\alpha(s_2^{(m)})$ are in different sides of plane P. This holds for all $m \in \mathbb{N}$, so as $m \to \infty$, $p \equiv \alpha(s) = \lim_{m\to\infty} \alpha(s_1^{(m)})$ lies on the left side of P, and $p \equiv \alpha(s) = \lim_{m\to\infty} \alpha(s_2^{(m)})$ lies on the right side of P, hence $p = \alpha(s) \in P$. Since P contains $\alpha(s)$ and has $b(s)$ as a normal vector, P is the osculating plane of α at s.

Problem 4 (Do Carmo 1.6.2)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that

*__a.__ The osculating plane at s is the limit position of the plane passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \to 0$.

b. The limit position of the circle passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \to 0$ is a circle in the osculating plane at s, the center of which is on the line that contains $n(s)$ and the radius of which is the radius of curvature $1/k(s)$; this circle is called the <u>osculating circle</u> at s.

Solution 4.

(a) Since the plane, which we will call P, by construction passes through $\alpha(s)$, we are left to show that the normal vector n_P of P converges to $b(s)$ in the limit $h_1, h_2 \to 0$. We have

$$\begin{aligned} n_P &= \frac{(\alpha(s+h_1) - \alpha(s)) \wedge (\alpha(s+h_2) - \alpha(s))}{|(\alpha(s+h_1) - \alpha(s)) \wedge (\alpha(s+h_2) - \alpha(s))|} \\ &= \frac{\left(h_1\alpha'(s) + O(h_1^2)\right) \wedge \left(h_2\alpha'(s) + O(h_2^2)\right)}{|(h_1\alpha'(s) + O(h_1^2)) \wedge (h_2\alpha'(s) + O(h_2^2))|} \\ &= \left(\frac{\alpha'(s) \wedge \alpha''(s)}{|\alpha'(s) \wedge \alpha''(s)|} + O(h_1) + O(h_2)\right), \end{aligned}$$

hence

$$\lim_{h_1,h_2\to 0} n_P = \frac{\alpha'(s) \wedge \alpha''(s)}{|\alpha'(s) \wedge \alpha''(s)|}.$$

Then the binormal vector is parallel to N_P since

$$b(s) = t(s) \wedge n(s) = \alpha'(s) \wedge \alpha''(s)/|\alpha''(s)| \parallel n_P.$$

(b) Without loss of generality, shift the origin to s so that $\alpha(s), \alpha(s + h_1), \alpha(s + h_2)$ become $\alpha(0), \alpha(h_1), \alpha(h_2)$, respectively. Let (x_0, y_0, z_0) be the center of the circle passing through $\alpha(0)$, $\alpha(h_1)$, and $\alpha(h_2)$, then the equation of the circle can be written as $F(s) = (x(s) - x_0)^2 + (y(s) - y_0)^2 + (z(s) - z_0)^2 - r^2$. Calculate the derivatives to be

$$F'(s) = 2(x(s) - x_0)x'(s) + 2(y(s) - y_0)y'(s) + 2(z(s) - z_0)z'(s)$$

and

$$\begin{aligned} F''(s) = {} & 2(x'(s))^2 + 2(y'(s))^2 + 2(z'(s))^2 \\ & + 2(x(s) - x_0)x''(s) + 2(y(s) - y_0)y''(s) + 2(z(s) - z_0)z''(s). \end{aligned}$$

Taking the limit as $s \to 0$ gives $F'(0) = -2x_0$ and $F''(0) = 2 - 2k(0)y_0$. Since the plane passes through $\alpha(0), \alpha(h_1), \alpha(h_2)$, we have $F(0) = F(h_1) = F(h_2) = 0$. By the Mean Value Theorem, there exists some $s_1 \in (0, h_1)$ such that $F'(s_1) = 0$. As $h_1 \to 0$, we have $s_1 \to 0$, by continuity of F we have $F'(s_1) \to 0$ as $s_1 \to 0$ as $h_1, h_2 \to 0$. Similarly, suppose $h_1 < h_2$, there exists some $s_2 \in (h_1, h_2)$ such that $F'(s_2) = 0$. By the Mean Value Theorem, there exists some $s_3 \in (s_1, s_2)$ such that $F''(s_3) = 0$. As $h_1, h_2 \to 0$, we have $s_1, s_2 \to 0$, so by continuity of F'', $F''(s_3) \to 0$ as $s_3 \to 0$. Therefore,

$$\lim_{h_1,h_2\to 0} F'(s_1) = F'(0) = -2x_0 = 0 \implies x_0 = 0,$$

and

$$\lim_{h_1,h_2\to 0} F''(s_2) = F''(0) = 2 - 2k(0)y_0 = 0 \implies y_0 = \frac{1}{k(0)}.$$

By (a) we know the circle lies on the osculating plane at $\alpha(0)$ as $h_1, h_2 \to 0$, so $c \to 0$. Hence the center of the circle converges to $(0, 1/k(0), 0)$, which lies on the line containing $n(0)$, and the radius converges to $1/k(0)$.