2025 Fall Introduction to Geometry

Homework 4 (Due Oct 3, 2025)

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Definition 1 (Regular surface)**.** A subset $S \subseteq \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood $V \subseteq \mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ such that

(i) $\mathbf{x}$ is (infinitely) differentiable.

(ii) $\mathbf{x}$ is a homeomorphism, i.e. $\mathbf{x}$ is a bijection, and both $\mathbf{x}$ and $\mathbf{x}^{-1}$ are continuous.

(iii) For each $q \in U$, the differential $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (the regularity condition).

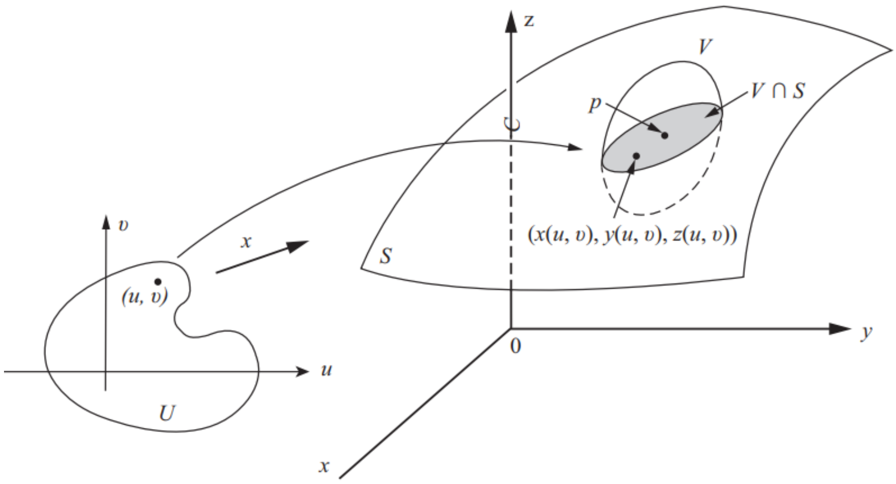

Figure 2-1

Definition 2 (Differentiability on regular surfaces)**.** Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$. f is differentiable in V if it is differentiable at all points of V.

Problem 1 (Do Carmo 2.2.11)**.** Show that the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \ ; \ z = x^2 - y^2\}$$

is a regular surface and check that parts (a) and (b) are parametrizations for S:

(a) $\mathbf{x}(u, v) = (u + v,\ u - v,\ 4uv), \quad (u, v) \in \mathbb{R}^2.$

(b) $\mathbf{x}(u, v) = (u \cosh v,\ u \sinh v,\ u^2), \quad (u, v) \in \mathbb{R}^2,\ u \neq 0.$

Which parts of S do these parametrizations cover?

Solution 1.

Notice that $z(x, y) = x^2 - y^2$ is a differentiable function from the open set $U = \mathbb{R}^2$ to $\mathbb{R}$, so by Proposition 2.2.1 in Do Carmo, S is a regular surface. Recall that a map $\mathbf{x} : U \to V \cap S$ if $\mathbf{x}$ is differentiable, a homeomorphism, and $\mathrm{d}\mathbf{x}_p$ is one-to-one for all $p \in U$.

(a) The map $\mathbf{x}$ is a polyminial in u and v, so it is differentiable. By explicit calculation,

$$\mathrm{d}\mathbf{x}_q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{pmatrix}$$

in the canonical basis, so $|\partial(x, y)/\partial(u, v)| = 2$ and $\mathrm{d}\mathbf{x}$ is one-to-one. To show that $\mathbf{x}$ is a homeomorphism, observe that for any $(x, y, z) \in S$, we have $z = x^2 - y^2$, so $z = (u + v)^2 - (u - v)^2 = 4uv$, and

$$u = \frac{x + y}{2}, \quad v = \frac{x - y}{2}$$

from the remaining equations. This determines a unique (u, v) for each $(x, y, z) \in S$, and we can conclude that the inverse map $\mathbf{x}^{-1}$ exists and is continuous.

(b) The map $\mathbf{x}$ is a composition of polynomials and exponential functions, so it is differentiable. By explicit calculation,

$$\mathrm{d}\mathbf{x}_q = \begin{pmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \\ 2u & 0 \end{pmatrix}$$

in the canonical basis, so $|\partial(x, y)/\partial(u, v)| = u$, and $\mathrm{d}\mathbf{x}$ is one-to-one for $u \neq 0$. To show that $\mathbf{x}$ is a homeomorphism, observe that for any $(x, y, z) \in S$ with $x^2 - y^2 > 0$, we have $z = x^2 - y^2$, so $z = u^2(\cosh^2 v - \sinh^2 v) = u^2$, and

$$u = \pm\sqrt{x^2 - y^2}, \quad v = \tanh^{-1} \frac{y}{x}$$

from the remaining equations. This determines a unique (u, v) for each $(x, y, z) \in S$ with $x^2 - y^2 > 0$, and we can conclude that the inverse map $\mathbf{x}^{-1}$ exists and is continuous.

Parametrization (a) covers the whole surface S, while parametrization (b) only covers the parts of S where $|x| > |y|$.

Remark. The graph of $z = f(x, y) = x^2 - y^2$ is a hyperbolic paraboloid, also known as saddle, shown in figure 1, 1.

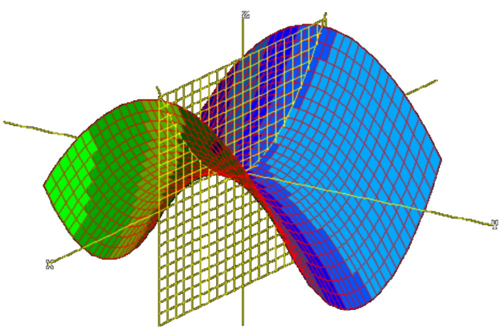

Figure 1: XZ plane projection

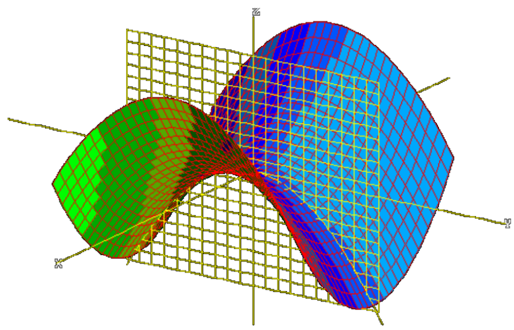

Figure 2: YZ plane projection

Problem 2 (Do Carmo 2.2.16)**.** One way to define a system of coordinates for the sphere S^2, given by

$$x^2 + y^2 + (z - 1)^2 = 1,$$

is to consider the so-called stereographic projection

$$\pi : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$$

which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy-plane with the straight line which connects N to p (Fig. 2–12). Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 \setminus \{N\}$ and (u, v) lies in the xy-plane.

a. Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$x = \frac{4u}{u^2 + v^2 + 4}, \qquad y = \frac{4v}{u^2 + v^2 + 4}, \qquad z = \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}.$$

b. Show that it is possible, using stereographic projection, to cover the sphere with two coordinate neighborhoods.

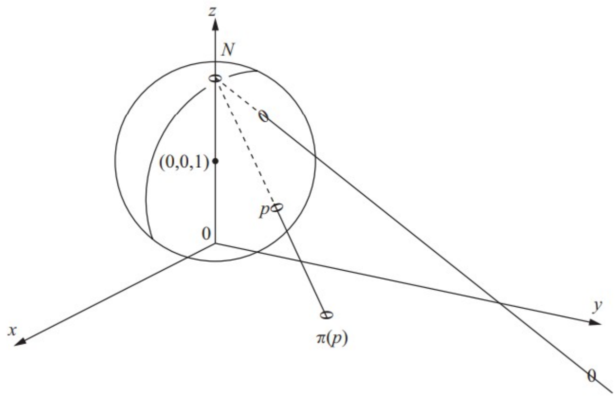

Figure 2-12. The stereographic projection.

Solution 2.

a. Let's construct the map $\pi : S^2 \to \mathbb{R}^2$ explicitly. For a point $p = (x, y, z) \in S^2 \setminus \{N\}$, the line connecting N and p can be parametrized as

$$L(t) = N + t(p - N) = (0, 0, 2) + t(x, y, z - 2) = (tx, ty, 2 + t(z - 2)) \tag{1}$$

The intersection of this line with the xy-plane occurs when $z = 0$, so $t = 2/(2 - z)$. Substituting this back to equation (1) gives

$$\pi(p) = (u, v) = \left(\frac{2x}{2 - z}, \frac{2y}{2 - z}\right).$$

Solving for (x, y) gives

$$(x, y) = \left(\frac{u(2 - z)}{2}, \frac{v(2 - z)}{2}\right).$$

From the equation for the sphere, we have

$$\left(\frac{u(2 - z)}{2}\right)^2 + \left(\frac{v(2 - z)}{2}\right)^2 + (z - 1)^2 = 1 \implies z = \frac{2(u^2 + v^2)}{u^2 + v^2 + 4},$$

hence

$$x = \frac{4u}{u^2 + v^2 + 4}, \quad y = \frac{4v}{u^2 + v^2 + 4}, \quad z = \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}.$$

b. Using the inverse stereographic projection π^{-1}, we can cover the whole sphere except the north pole N. To cover the north pole, use another stereographic projection from the south pole $S = (0, 0, 0)$ to the xy-plane, with the inverse map given by

$$x = \frac{4u}{u^2 + v^2 + 4}, \quad y = \frac{4v}{u^2 + v^2 + 4}, \quad z = \frac{8}{u^2 + v^2 + 4}.$$

Problem 3 (Do Carmo 2.2.19*)**.**

Let $\alpha : (-3, 0) \to \mathbb{R}^2$ be defined by (Fig. 2–13)

$$\alpha(t) = \begin{cases} (0,\ -(t+2)), & t \in (-3, -1), \\ \text{a regular parametrized curve joining } p = (0, -1) \text{ to } q = \left(\dfrac{1}{\pi}, 0\right), & t \in (-1, -\frac{1}{\pi}), \\ (-t,\ \sin \frac{1}{t}), & t \in \left(-\frac{1}{\pi}, 0\right). \end{cases}$$

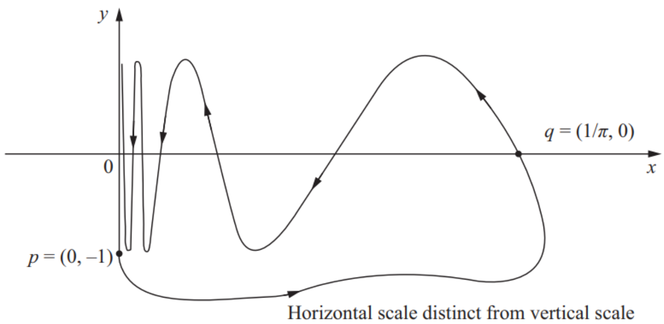

Figure 2-13

It is possible to define the curve joining p to q so that all the derivatives of α are continuous at the corresponding points and α has no self-intersections. Let C be the trace of α.

a. Is C a regular curve?

b. Let a normal line to the plane $\mathbb{R}^2$ run through C so that it describes a "cylinder" S. Is S a regular surface?

Solution 3.

a. Let C be the trace of α, α is said to be regular if at every point $p \in C$, C is the graph of a C^1 function $y = f(x)$ or $x = g(y)$ in a neighborhood of p. Notice that the origin $(0, 0)$ belongs to the trace of α since $\alpha(-2) = (0, 0)$. Consider the sequence $t_n = -\frac{1}{2n\pi}$, which satisfies $t_n \in (-\frac{1}{\pi}, 0)$ for all $n \in \mathbb{N}$. Therefore, in any neighborhood of $(0, 0)$, we can find some $n \in \mathbb{N}$ such that $\alpha(t_n) \in U$, so C cannot be the graph of $x = f(y)$ locally. Similarly, C cannot be the graph of $y = g(x)$ on the line segment $\{0\} \times (-1, 1) \subseteq \mathbb{R}^2$. Hence, C is not a regular curve.

b. If the surface S were regular, then by Do Carmo Proposition 2.2.3, there exists a neighborhood V of any $p \in S$ such that V is the graph of a differentiable function $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$. However, consider a point $p \in (-\frac{1}{\pi}, 0, z)$ on the side boundary of S. In (a) we concluded that locally around $(0, 0, z)$, the curve (translated by some z along the z axis) is not the graph of a C^1 function $x = g(y, z)$ or $y = h(x, z)$, while z cannot be a function of x, y. Therefore, S is not a regular surface.

Problem 4 (Do Carmo 2.3.5*)**.** Let $S \subset \mathbb{R}^3$ be a regular surface, and let $d : S \to \mathbb{R}$ be given by

$$d(p) = \|p - p_0\|,$$

where $p \in S$, $p_0 \in \mathbb{R}^3$, and $p_0 \notin S$; that is, d is the distance from p to a fixed point p_0 not in S. Prove that d is differentiable.

Solution 4. By definition 2, it suffices to show that for any parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, the composition $d \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable. Since S is a regular surface, for any point $p \in S$, there exists a neighborhood $V \subseteq \mathbb{R}^3$ of p such that $V \cap S$ is the graph of a differentiable function $z(x, y)$ or $x(y, z)$ or $y(x, z)$. Assume that $V \cap S$ is the graph of a differentiable function $z(x, y)$, then define a parametrization

$$\mathbf{x}(u, v) = (u, v, z(u, v)), \quad (u, v) \in U \subseteq \mathbb{R}^2,$$

where U is open in $\mathbb{R}^2$. The composition $d \circ \mathbf{x} : U \to \mathbb{R}$ is given by

$$\begin{aligned}(d \circ \mathbf{x})(u, v) = d(\mathbf{x}(u, v)) &= \sqrt{\langle \mathbf{x}(u, v) - p_0, \mathbf{x}(u, v) - p_0 \rangle} \\ &= \sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u, v) - z_0)^2}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial}{\partial u}(d \circ \mathbf{x})(u, v)\bigg|_{(u,v)} &= \frac{(u - x_0 + (z(u, v) - z_0)z_u(u, v))}{\sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u, v) - z_0)^2}}, \\ \frac{\partial}{\partial v}(d \circ \mathbf{x})(u, v)\bigg|_{(u,v)} &= \frac{(v - y_0 + (z(u, v) - z_0)z_v(u, v))}{\sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u, v) - z_0)^2}},\end{aligned}$$

and $z(u, v)$ is differentiable, we conclude that $d \circ \mathbf{x}$ is differentiable except when $(u, v) = (x_0, y_0) = \mathbf{x}^{-1}(p_0)$. Since the choice of $p \in S$ is arbitrary, we conclude that d is differentiable on $S \backslash \{p_0\}$.

Problem 5 (Do Carmo 2.3.10)**.** Let C be a plane regular curve which lies on one side of a straight line r of the plane and meets r at the points p, q (Fig. 2–21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

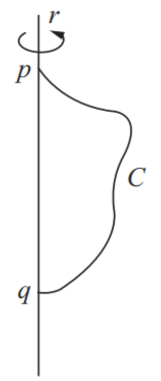

Figure 2-21

Solution 5. We can analyze the point $p \in C$ locally. Assume that r is the z axis, and C is the graph of a differentiable function $y = f(x)$ in a neighborhood of p, since C is a regular curve. Since S is the surface of revolution generated by rotating C about r, we claim that there is a local chart at $p \in S$ given by

$$\mathbf{x} : U \subseteq \mathbb{R}^2 \to S, \quad (x, y) \mapsto (x, y, f(\sqrt{x^2 + y^2})),$$

where U is an open set in $\mathbb{R}^2$. We will check each condition given in definition (1) for S.

(i) $\mathbf{x}$ is differentiable. We can calculate its differential at some $(x, y) \in U$ as

$$\mathrm{d}\mathbf{x}_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{x}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) & \dfrac{y}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) \end{pmatrix}. \tag{2}$$

Since f is differentiable, the partial derivatives of $\mathbf{x}$ exist whenever $(x, y) \neq (0, 0)$. By symmetry, $f(w) = f(-w)$, so $f'(w) = -f'(-w)$. When $(x, y) = (0, 0)$, we have $f'(0) = 0$, and

$$\frac{x}{\sqrt{x^2+y^2}}, \quad \frac{y}{\sqrt{x^2+y^2}}$$

are bounded, so $\mathrm{d}\mathbf{x}_{(x,y)}$ exists at $(0, 0)$. To satisfy the symmetry condition, we require that f' is odd, hence f is even, and all the odd-order derivatives of f vanish at 0. Similarly, the odd-order derivatives of g such that $y = g(x)$ in a neighborhood of q must also vanish.

(ii) $\mathbf{x}$ is a homeomorphism, since the graph of a continuous function is homeomorphic to its domain.

(iii) From equation (2), we have $|\partial(x, y)/\partial(u, v)| = 1$, so $\mathrm{d}\mathbf{x}$ is one-to-one. Hence $\mathrm{d}\mathbf{x}_{(x,y)}$ is one-to-one for all $(x, y) \in U$.