2025 Fall Introduction to Geometry

Homework 5 (Due Oct 10, 2025)

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October 17, 2025

Problem 1 (Do Carmo 2.3.16*)**.** Let $R^2 = \{(x, y, z) \in \mathbb{R}^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P : \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \quad a_0 \neq 0,\ a_i \in \mathbb{C},\ i = 0, \ldots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F : S^2 \to S^2$ given by

$$F(p) = \begin{cases} \pi_N^{-1} \circ P \circ \pi_N(p), & \text{if } p \in S^2 - \{N\}, \\ N, & \text{if } p = N, \end{cases}$$

is differentiable.

Solution 1. Given a point $p \in S^2 \backslash \{N\}$, write it as $p = (x, y, z)$. Since the composition of differentiable functions is differentiable, we only need to show that π_N, π_N^{-1} and P are differentiable. The stereographic projection $\pi_N : S^2 \backslash \{N\} \to \mathbb{R}^2$ is given by

$$\pi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Since $z \neq 1$ for all $p \in S^2 \backslash \{N\}$, π_N is differentiable. Similarly, note that the inverse stereographic projection $\pi_N^{-1} : \mathbb{R}^2 \to S^2 \backslash \{N\}$ is given by

$$\pi_N^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

Since $u^2 + v^2 + 1 > 0$ for all $(u, v) \in \mathbb{R}^2$, π_N^{-1} is differentiable. Moreover, polynomials are differentiable everywhere, so P is differentiable. Thus, F is differentiable on $S^2 \backslash \{N\}$.

Problem 2 (Do Carmo 2.4.10. Tubular Surfaces)**.** Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r\big(n(s)\cos v + b(s)\sin v\big), \quad r = \text{const.} \neq 0,\ s \in I,$$

be a parametrized surface (the tube of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

$$N(s, v) = -\big(n(s)\cos v + b(s)\sin v\big).$$

Solution 2. Let $\mathbf{x} : U \to \mathbb{R}^3$ as defined in the problem statement be a regular Parametrization, where U is an open set in $\mathbb{R}^2$. The unit normal vector at each point $q \in \mathbf{x}(U)$ is defined as

$$N(q) = \frac{\mathbf{x}_s \wedge \mathbf{x}_v}{|\mathbf{x}_s \wedge \mathbf{x}_v|}(q).$$

Let prime denote derivative with respect to s. Then we have

$$\mathbf{x}_s = \alpha'(s) + r\big(n'(s)\cos v + b'(s)\sin v\big), \quad \mathbf{x}_v = r\big(-n(s)\sin v + b(s)\cos v\big),$$

and by the Frenet-Serret formulas,

$$\alpha'(s) = t(s), \quad n'(s) = -\kappa(s)t(s) - \tau(s)b(s), \quad b'(s) = \tau(s)n(s),$$

where t is the unit tangent, κ is the curvature, and τ is the torsion of α. Thus,

$$\begin{aligned}
\mathbf{x}_s &= t(s) + r\big((-\kappa(s)t(s) - \tau(s)b(s))\cos v + \tau(s)n(s)\sin v\big),\\
\mathbf{x}_v &= r\big(-n(s)\sin v + b(s)\cos v\big).
\end{aligned}$$

Now suppress s and compute the wedge product in the Frenet frame $\{t, n, b\}$:

$$\begin{aligned}
\mathbf{x}_s \wedge \mathbf{x}_v &= \big(t + r\big(-\kappa t\cos v - \tau b\cos v + \tau n\sin v\big)\big) \wedge r\big(-n\sin v + b\cos v\big)\\
&= -r(t\wedge n)\sin v + r(t\wedge b)\cos v - r^2\kappa\sin v\cos v(t\wedge n) - r^2\kappa\cos^2 v(t\wedge b)\\
&\quad + r^2\tau\sin v\cos v(b\wedge n) + r^2\tau\sin v\cos v(n\wedge b)\\
&= -r(1 - r\kappa\cos v)\,(\cos v n + \sin v b)\,.
\end{aligned}$$

Dividing by the norm and noting that n and b are unit length and orthogonal, we have

$$N(s, v) = -\big(n(s)\cos v + b(s)\sin v\big).$$

Problem 3 (Do Carmo 2.4.17)**.** Two regular surfaces S_1 and S_2 intersect transversally if whenever $p \in S_1 \cap S_2$ then $T_p(S_1) \neq T_p(S_2)$. Prove that if S_1 intersects S_2 transversally, then $S_1 \cap S_2$ is a regular curve.

Solution 3. Let S_1, S_2 be two regular surfaces that intersect transversally, and let $p \in S_1 \cap S_2$. Since S_1, S_2 are regular surfaces, there exists a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ and a neighborhood V_1 of p such that $S_1 \cap V_1 = f^{-1}(0) \cap V_1$. Similarly, there exists a differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$ and a neighborhood V_2 of p such that $S_2 \cap V_2 = g^{-1}(0) \cap V_2$. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $F(q) = (f(q), g(q))$. Then

$$F^{-1}(0,0) = f^{-1}((0,0)) \cap g^{-1}((0,0)) \supseteq (V_1 \cap V_2) \cap (S_1 \cap S_2).$$

Let $V = V_1 \cap V_2$. In V, we have $S_1 \cap S_2 = F^{-1}(0,0)$. Since $T_p(S_1) \neq T_p(S_2)$, we have $N_{p_1}(0,0) \wedge N_{p_2}(0,0) \neq 0$, where

$$N_{p_1} = \frac{(f_x, f_y, f_z)(p)}{\|(f_x, f_y, f_z)(p)\|}, \quad N_{p_2} = \frac{(g_x, g_y, g_z)(p)}{\|(g_x, g_y, g_z)(p)\|}.$$

Hence

$$\mathrm{d}F_{(x,y,z)} = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}(x, y, z) \neq 0,$$

and $\mathrm{d}F$ is surjective. Therefore, $(0,0)$ is a regular point of F, and by [Do Carmo] Problem 2.2.17 (b) (The inverse image of a regular value of a differentiable map $F : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ is a regular curve in $\mathbb{R}^3$), $S_1 \cap S_2$ is a regular curve.

Problem 4 (Do Carmo 2.4.23)**.** Prove that the map $F : S^2 \to S^2$ defined in Exercise 16 of Sec. 2-3 has only a finite number of critical points (see Exercise 13).

Solution 4. From Problem 2.3.16, $F : S^2 \to S^2$ is differentiable. Let $p \in S^2$ be a critical point of F, then $\mathrm{d}F_p = 0$. Since $F = \pi_N^{-1} \circ P \circ \pi_N$, by the chain rule, we have

$$\mathrm{d}F_p = \mathrm{d}(\pi_N^{-1})_{P(\pi_N(p))} \circ \mathrm{d}P_{\pi_N(p)} \circ \mathrm{d}(\pi_N)_p.$$

Note that $\mathrm{d}(\pi_N)_p$ and $\mathrm{d}(\pi_N^{-1})_{P(\pi_N(p))}$ are isomorphisms, so $\mathrm{d}F_p = 0$ if and only if $\mathrm{d}P_{\pi_N(p)} = 0$. Since $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n, $\mathrm{d}P$ is a polynomial of degree $n-1$, and thus has $n-1$ roots by the Fundamental Theorem of Algebra. Therefore, the map $F : S^2 \to S^2$ has only a finite number of critical points.

Problem 5 (Do Carmo 2.4.28)**.**

a. Define regular value for a differentiable function $f : S \to \mathbb{R}$ on a regular surface S.

b. Show that the inverse image of a regular value of a differentiable function on a regular surface S is a regular curve on S.

Solution 5.

a. A regular value of a differentiable function $f : S \to \mathbb{R}$ defined on a regular surface S is a value $c \in \mathbb{R}$ such that for every point $p \in f^{-1}(c)$, the differential $\mathrm{d}f_p : T_p(S) \to \mathbb{R}$ is surjective (i.e., $\mathrm{d}f_p \neq 0$).

b. Let c be a regular value of a differentiable function $f : S \to \mathbb{R}$ and let $p \in f^{-1}(c)$. Pick a local parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ such that $\mathbf{x}((0,0)) = p$. Define $g : U \to \mathbb{R}$ by $g = f \circ \mathbf{x}$, then $g(0,0) = f(\mathbf{x}(0,0)) = f(p) = c$. Since $\mathrm{d}f_p \neq 0$ and $\mathrm{d}\mathbf{x}_{(0,0)}$ is surjective onto T_pS, we have $\mathrm{d}g_{(0,0)} \neq 0$. By the Implicit Function Theorem, there exists a neighborhood $V \subseteq U$ of $(0,0)$ such that $g^{-1}(c) \cap V$ is the graph of a C^1 function, say $v = \phi(u)$. Then we can define a local parametrization of the curve $f^{-1}(c)$ on S by

$$\alpha(u) = \mathbf{x}(u, \phi(u)), \quad u \in I$$

where I is some neighborhood of $u = 0$. Suppose for some u^*, we have $\alpha'(u^*) = 0$, then

$$\mathrm{d}\mathbf{x}_{(u^*, \phi(u^*))}\,(1, \phi'(u^*)) = 0.$$

Since $\mathrm{d}\mathbf{x}$ is one-to-one, we must have $(1, \phi'(u^*)) = 0$, contradiction. Thus, $\alpha'(u) \neq 0$ for all $u \in I$, and in a neighborhood of each $p \in f^{-1}(c)$, $f^{-1}(c)$ is the image of a regular curve α on S. Patching the local parametrizations together, we conclude that $f^{-1}(c)$ is a regular curve on S.