2025 Fall Introduction to Geometry

Homework 6 (Due October 17, 2025)

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Problem 1 (Do Carmo 2.5.3). Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2–2).

Solution 1. Refer to Exercise 2.2.16, let the sphere be $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 1\}$. The stereographic projection from the north pole $N = (0, 0, 2)$ to the xy-plane is given by

$$\mathbf{x}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

We have

$$\begin{aligned} \mathbf{x}_u &= \left(\frac{4(-u^2 + v^2 + 4)}{(u^2 + v^2 + 4)^2}, \frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{16u}{(u^2 + v^2 + 4)^2} \right), \\ \mathbf{x}_v &= \left(\frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{4(u^2 - v^2 + 4)}{(u^2 + v^2 + 4)^2}, \frac{16v}{(u^2 + v^2 + 4)^2} \right). \end{aligned}$$

Thus we have

$$\begin{aligned} E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle &= \frac{16(-u^2 + v^2 + 4)^2 + 64u^2v^2 + 256u^2}{(u^2 + v^2 + 4)^4} = \frac{16}{(u^2 + v^2 + 4)^2}, \\ F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle &= \frac{-32uv(-u^2 + v^2 + 4) - 32uv(u^2 - v^2 + 4) + 256uv}{(u^2 + v^2 + 4)^4} = 0, \\ G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle &= \frac{64u^2v^2 + 16(u^2 - v^2 + 4)^2 + 256v^2}{(u^2 + v^2 + 4)^4} = \frac{16}{(u^2 + v^2 + 4)^2}. \end{aligned}$$

Therefore, the first fundamental form is

$$I_p = \frac{16}{(u^2 + v^2 + 4)^2} \left((u')^2 + (v')^2 \right).$$

Problem 2 (Do Carmo 2.5.7). The coordinate curves of a parametrization $\mathbf{x}(u, v)$ constitute a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

Solution 2. The coordinate curves of a parametrization $\mathbf{x}(u, v)$ are the curves obtained by fixing one of the parameters and varying the other. Suppose we have a quadrilateral formed by the coordinate curves at points $(u_1, v_1), (u_2, v_1), (u_2, v_2), (u_1, v_2)$. Let $s(\mathbf{x}(u_1, v_1), \mathbf{x}(u_2, v_2)) \equiv s((u_1, v_1), (u_2, v_2))$ denote the arc length between two points. Then the lengths of the opposite sides are equal if and only if

$$\begin{aligned} s((u_1, v_1), (u_2, v_1)) = s((u_1, v_2), (u_2, v_2)) &\implies \int_{u_1}^{u_2} \mathrm{d}u \, \sqrt{E(u, v_1)} = \int_{u_1}^{u_2} \mathrm{d}u \, \sqrt{E(u, v_2)}, \\ s((u_1, v_1), (u_1, v_2)) = s((u_2, v_1), (u_2, v_2)) &\implies \int_{v_1}^{v_2} \mathrm{d}v \, \sqrt{G(u_1, v)} = \int_{v_1}^{v_2} \mathrm{d}v \, \sqrt{G(u_2, v)}. \end{aligned}$$

Since u_1, u_2, v_1, v_2 are arbitrary, we have

$$\sqrt{E(u, v_1)} = \sqrt{E(u, v_2)}, \quad \sqrt{G(u_1, v)} = \sqrt{G(u_2, v)}.$$

Therefore, E is independent of v and G is independent of u, giving the desired result:

$$\frac{\partial E}{\partial v} = 0, \quad \frac{\partial G}{\partial u} = 0.$$

Problem 3 (Do Carmo 2.5.10)**.** Let $P = \{(x, y, z) \in \mathbb{R}^3;\ z = 0\}$ be the xy-plane and let $\mathbf{x}\colon U \to P$ be a parametrization of P given by

$$\mathbf{x}(\rho, \theta) = (\rho\cos\theta,\ \rho\sin\theta),$$

where

$$U = \{(\rho, \theta) \in \mathbb{R}^2;\ \rho > 0,\ 0 < \theta < 2\pi\}.$$

Compute the coefficients of the first fundamental form of P in this parametrization.

Solution 3. We have

$$\mathbf{x}_\rho = (\cos\theta, \sin\theta, 0), \tag{1}$$
$$\mathbf{x}_\theta = (-\rho\sin\theta, \rho\cos\theta, 0). \tag{2}$$

Thus the coefficients of the first fundamental form are

$$\begin{aligned} E &= \langle \mathbf{x}_\rho, \mathbf{x}_\rho \rangle = \cos^2\theta + \sin^2\theta = 1, \\ F &= \langle \mathbf{x}_\rho, \mathbf{x}_\theta \rangle = -\rho\cos\theta\sin\theta + \rho\sin\theta\cos\theta = 0, \\ G &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle = \rho^2\sin^2\theta + \rho^2\cos^2\theta = \rho^2. \end{aligned}$$

Problem 4 (Do Carmo 2.5.15, Orthogonal Families of Curves)**.**

a. Let E, F, G be the coefficients of the first fundamental form of a regular surface S in the parametrization $\mathbf{x}\colon U \subset \mathbb{R}^2 \to S$. Let $\varphi(u, v) = \text{const.}$ and $\psi(u, v) = \text{const.}$ be two families of regular curves on $\mathbf{x}(U) \subset S$ (cf. Exercise 28, Sec. 2–4). Prove that these two families are orthogonal (i.e., whenever two curves of distinct families meet, their tangent lines are orthogonal) if and only if

$$E\,\varphi_v\psi_v \ - \ F\big(\varphi_u\psi_v + \varphi_v\psi_u\big) \ + \ G\,\varphi_u\psi_u \ = \ 0.$$

b. Apply part a to show that on the coordinate neighborhood $\mathbf{x}(U)$ of the helicoid of Example 3 the two families of regular curves

$$v\cos u = \text{const.}, \quad v \neq 0, \qquad \big(v^2 + a^2\big)\sin^2 u = \text{const.}, \quad v \neq 0, \quad u \neq \pi,$$

are orthogonal.

Solution 4.

a. Suppose $\varphi(u, v) = c_1$ and $\psi(u, v) = c_2$ are two families of regular curves on $\mathbf{x}(U) \subset S$. The curves satisfy

$$\phi_u u' + \phi_v v' = 0, \quad \psi_u u' + \psi_v v' = 0.$$

So we can choose the tangent vectors of the two families of curves to be

$$t(\varphi) = -\varphi_v\mathbf{x}_u + \varphi_u\mathbf{x}_v, \quad t(\psi) = -\psi_v\mathbf{x}_u + \psi_u\mathbf{x}_v.$$

The two families are orthogonal if and only if $\langle t(\varphi), t(\psi)\rangle = 0$, which is equivalent to

$$\begin{aligned}\langle t(\varphi), t(\psi)\rangle &= \langle -\varphi_v \mathbf{x}_u + \varphi_u \mathbf{x}_v, -\psi_v \mathbf{x}_u + \psi_u \mathbf{x}_v\rangle \\ &= \varphi_v\psi_v\langle \mathbf{x}_u, \mathbf{x}_u\rangle - \varphi_v\psi_u\langle \mathbf{x}_u, \mathbf{x}_v\rangle - \varphi_u\psi_v\langle \mathbf{x}_v, \mathbf{x}_u\rangle + \varphi_u\psi_u\langle \mathbf{x}_v, \mathbf{x}_v\rangle \\ &= E\varphi_v\psi_v - F(\varphi_u\psi_v + \varphi_v\psi_u) + G\varphi_u\psi_u = 0.\end{aligned}$$

b. From Example 3, the helicoid is given by the parametrization $\mathbf{x}(u, v) = (v\cos u, v\sin u, au)$, with the coefficients of the first fundamental form being

$$E(u, v) = v^2 + a^2, \quad F(u, v) = 0, \quad G(u, v) = 1.$$

Let $\phi(u, v) = v\cos u$, $\psi(u, v) = (v^2 + a^2)\sin^2 u$. Then

$$\begin{aligned}\phi_u &= -v\sin u, & \phi_v &= \cos u, \\ \psi_u &= 2(v^2 + a^2)\sin u\cos u, & \psi_v &= 2v\sin^2 u.\end{aligned}$$

Substituting these into equation (**??**) in part (a), we have

$$(v^2 + a^2)\cos u(2v\sin^2 u) - 0 + 1(-v\sin u)(2(v^2 + a^2)\sin u\cos u) = 0.$$

Therefore, the two families of regular curves are orthogonal.