2025 Fall Introduction to Geometry

Homework 7 (Due Nov 7, 2025)

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Exercise 1 (Do Carmo 3.2.2)**.** Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Solution 1. Suppose a surface S is tangent to a plane Π along a curve C. Let $p \in C$ be an arbitrary point on the curve. Parametrize the curve C by $\alpha : I \to S \cap \Pi$, where I is an open interval containing 0 and $\alpha(0) = p$. Let $N : S \to S^2$ be the Gauss map of S. Since the tangent plane of S is Π for all $p \in S$, the unit normal $N(\alpha(s))$ is equal to the constant normal n of Π. Thus,

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} N(\alpha(s)) = \mathrm{d}N_{\alpha(s)}(\alpha'(s)).$$

Therefore, the differential of the Gauss map $\mathbf{d}N_p$ has a nontrivial kernel containing $\alpha'(0) \neq 0$ for all $\alpha(s) \in S$. But $\mathbf{d}N_p : T_p(S) \to T_{N(p)}(S^2)$ is a linear map between finite-dimensional vector spaces, $\mathrm{d}N_p$ is not invertible, and hence $\det(\mathrm{d}N_p) \neq 0$ for all $p \in C$. Thus, all points on C are either parabolic or planar.

Exercise 2 (Do Carmo 3.2.8)**.** Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

- **a.** Paraboloid of revolution $z = x^2 + y^2$.
- **b.** Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.
- **c.** Catenoid $x^2 + y^2 = \cosh^2 z$.

Solution 2. Let's take the natural orientation: upward normal for graphs and outward normal for surfaces of revolution.

a. Let the graph be $z = f(x, y) = x^2 + y^2$, then the normal to the surface is

$$N = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}},$$

where $f_x = 2x$, $f_y = 2y$. Since $(x, y) \in \mathbb{R}^2$ and the z component $N^z = 1/\sqrt{1 + 4(x^2 + y^2)} \in (0, 1]$, the Gauss map is the open upper hemisphere of the unit sphere.

b. As a level set $F(x, y, z) = x^2 + y^2 - z^2 - 1$, the (outward) normal vector is

$$N = \frac{\nabla F}{|\nabla F|} = \frac{(2x, 2y, -2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x, y, -z)}{\sqrt{x^2 + y^2 + z^2}}.$$

Since $x^2 + y^2 = z^2 + 1 \geq 1$, the z component

$$N^z = -\frac{z}{\sqrt{x^2 + y^2 + z^2}} = -\frac{z}{\sqrt{2z^2 + 1}} \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

Thus, the Gauss map covers the open band $\{p \in S^2 \mid |N^z| < \frac{1}{\sqrt{2}}\}$.

c. Let's write this in the following parametrization:

$$\mathbf{x}(z,\theta) = (\cosh z\cos\theta, \cosh z\sin\theta, z), \quad z \in \mathbb{R}, \theta \in [0, 2\pi).$$

Then,

$$\mathbf{x}_z = (\sinh z\cos\theta, \sinh z\sin\theta, 1), \quad \mathbf{x}_\theta = (-\cosh z\sin\theta, \cosh z\cos\theta, 0).$$

The normal vector is given by

$$N = \frac{\mathbf{x}_z \times \mathbf{x}_\theta}{|\mathbf{x}_z \times \mathbf{x}_\theta|} = \frac{(-\cosh z\cos\theta, -\cosh z\sin\theta, \sinh z\cosh z)}{\sqrt{\cosh^2 z + \sinh^2 z\cosh^2 z}} = \frac{(-\cos\theta, -\sin\theta, \sinh z)}{\sqrt{1+\sinh^2 z}}.$$

$$\implies N = (-\operatorname{sech} z\cos\theta, -\operatorname{sech} z\sin\theta, \tanh z).$$

Since $\theta \in [0, 2\pi)$ and $N^z = -\tanh z \in (-1, 1)$, the spherical image $N(C) = S^2 \setminus \{(0,0,\pm 1)\}$.

Exercise 3 (Do Carmo 3.2.9)**.**

a. Prove that the image $N \circ \alpha$ by the Gauss map $N : S \to S^2$ of a parametrized regular curve $\alpha : I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere S^2 (called the spherical image of α).

b. If $C = \alpha(I)$ is a line of curvature, and k is its curvature at p, then

$$k = |k_n k_N|,$$

where k_n is the normal curvature at p along the tangent line of C and k_N is the curvature of the spherical image $N(C) \subset S^2$ at $N(p)$.

Solution 3.

a. Suppose $\alpha : I \to S$ is a parametrized regular curve with no planar or parabolic points. Then, the Gauss map $N : S \to S^2$ satisfies $\det(\mathrm{d}N_p) \neq 0$, and $\mathrm{d}N_p$ is invertible, and hence injective for all $p \in C$. Since α is a regular curve, $\alpha'(t) \neq 0$ for all $t \in I$, and hence

$$(N \circ \alpha)'(t) = \mathrm{d}N_{\alpha(t)}(\alpha'(t)) \neq 0,$$

which shows that the spherical image $N(C)$ is a regular curve on S^2.

b. Since C is a line of curvature, the tangent vector $t = \alpha'(s)$ at $p = \alpha(s)$ is a principal direction. Hence, $\mathcal{S}(t) = k_n t$ where k_n is the normal curvature along t at p. Let $N : S \to S^2$ be the Gauss map of S. Using $\mathrm{d}N = -\mathcal{S}(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} N(\alpha(s)) = \mathrm{d}N_{\alpha(s)}(\alpha'(s)) = -\mathcal{S}(t) = -k_n t.$$

Thus, $|N'| = |k_n|$, and the tangent vector of the spherical image $N(C)$ at $N(p)$ is

$$t_N = \frac{N'}{|N'|} = \frac{-k_n t}{|k_n|} = -\operatorname{sgn}(k_n) t.$$

Let s_N be the arc length parameter of the spherical image $N(C)$. Then,

$$|k_N| = \left|\frac{\mathrm{d}t_N}{\mathrm{d}s_N}\right| = \frac{|\mathrm{d}t_N/\mathrm{d}s|}{|\mathrm{d}s_N/\mathrm{d}s|} = \frac{\mathrm{d}t_N/\mathrm{d}s}{|N'|} = \frac{k}{|k_n|},$$

where we used $t' = kn$ in the last equality. Therefore, $k = |k_n k_N|$.

Exercise 4 (Do Carmo 3.2.10)**.** Assume that the osculating plane of a line of curvature $C \subset S$, which is nowhere tangent to an asymptotic direction, makes a constant angle with the tangent plane of S along C. Prove that C is a plane curve.

Solution 4. Let t, n, b be the Frenet frame of the curve C. Since the osculating plane makes a constant angle with the tangent plane of S, the unit normal N of S along C satisfies

$$b \cdot N = \text{const.}$$

Differentiate both sides with respect to the arc length parameter s of C and use Frenet's formula:

$$b' \cdot N + b \cdot N' = 0 \implies -\tau n \cdot N + b \cdot N' = 0.$$

Next, $N' = -\mathcal{S}(t)$ by the Weingarten formula, where $\mathcal{S}$ is the shape operator of S. Since C is a line of curvature, t is a principal direction of S, and $\mathcal{S}(t) = k_n t$, where k_n is the normal curvature of S along C. Thus,

$$-\tau n \cdot N - k_n b \cdot t = -\tau k_n / k = 0,$$

where k is the curvature of C. Since C is nowhere tangent to an asymptotic direction, $k_n \neq 0$, so $\tau = 0$. This implies $b' = -\tau n = 0$, so

$$\frac{\mathrm{d}}{\mathrm{d}s}(b \cdot c) = cb' = 0 \implies b = \text{const.}$$

and hence C is a plane curve.

Exercise 5 (*Do Carmo 3.2.14)**.** If the surface S_1 intersects the surface S_2 along the regular curve C, then the curvature k of C at $p \in C$ is given by

$$k^2 \sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta,$$

where λ_1 and λ_2 are the normal curvatures at p, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at p.

Solution 5. Suppose S_1 and S_2 intersect along the regular curve C. Let N_1, N_2 be the unit normals and let λ_1, λ_2 be the normal curvatures along the tangent line to C of S_1 and S_2, respectively. Let t, n, b be the Frenet frame of the curve C. Since C lies on S_1 and S_2, $t \perp N_i$, $i = 1, 2$. Thus, we can write $N_i = n\cos\phi_i + b\sin\phi_i$ for some $\phi_i \in [0, \frac{\pi}{2}]$, $i = 1, 2$. The normal curvatures are given by

$$\lambda_i = \alpha'' \cdot N_i = kn \cdot N_i = k\cos\phi_i, \quad i = 1, 2.$$

By definition, the angle θ between N_1 and N_2 satisfies

$$\cos\theta = N_1 \cdot N_2 = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2 = \cos(\phi_1 - \phi_2).$$

By direct computation, we have

$$\begin{aligned}
\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta &= k^2(\cos^2\phi_1 + \cos^2\phi_2 - 2\cos\phi_1\cos\phi_2\cos(\phi_1 - \phi_2)) \\
&= k^2\left(\cos^2\phi_1 + \cos^2\phi_2 - 2\cos\phi_1\cos\phi_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\right) \\
&= k^2\big(\cos^2\phi_1 + \cos^2\phi_2 - \cos^2\phi_1(1 - \sin^2\phi_2) \\
&\quad - \cos^2\phi_2(1 - \sin^2\phi_1) - 2\sin\phi_1\sin\phi_2\cos\phi_1\cos\phi_2\big) \\
&= k^2\left(\sin^2\phi_1\cos^2\phi_2 + \sin^2\phi_2\cos^2\phi_1 - 2\sin\phi_1\sin\phi_2\cos\phi_1\cos\phi_2\right) \\
&= k^2\sin^2(\phi_1 - \phi_2) = k^2\sin^2\theta.
\end{aligned}$$