2025 Fall Introduction to Geometry

Homework 8 (Due Nov 14, 2025)

物理三 黃紹凱 B12202004

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Definition 1 (Do Carmo 3.2.5, line of curvature)**.** If a regular connected curve $C \subseteq S$ is such that for all $p \in S$ the tangent line of C is a principal direction at p, then C is said to be a line of curvature of S.

Definition 2 (Do Carmo 3.2.9, asymptotic curve)**.** Let $p \in S$. An asymptotic direction of S at p is a direction in $T_p(S)$ for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subseteq S$ such that for each $p \in S$ the tangent line of C at p is an asymptotic direction.

Exercise 1 (Do Carmo 3.3.5, Enneper's Surface)**.** Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \qquad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

d. The lines of curvature are the coordinate curves.

e. The asymptotic curves are $u + v = \text{const.}$ and $u - v = \text{const.}$

Solution 1.

a. Calculate the first-order partial derivatives:

$$\mathbf{x}_u = \left(1 - u^2 + v^2,\ 2uv,\ 2u\right), \quad \mathbf{x}_v = \left(2uv,\ 1 - v^2 + u^2,\ -2v\right).$$

Then the coefficients of the first fundamental form are

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 = (1 + u^2 + v^2)^2, \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 2uv(1 - u^2 + v^2) + 2uv(1 + u^2 - v^2) - 4uv = 0, \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 4u^2v^2 + (1 + u^2 - v^2)^2 + 4v^2 = (1 + u^2 + v^2)^2.
\end{aligned}$$

b. Calculate the second-order partial derivatives:

$$\mathbf{x}_{uu} = (-2u,\, 2v,\, 2)\,, \quad \mathbf{x}_{uv} = (2v,\, 2u,\, 0)\,, \quad \mathbf{x}_{vv} = (2u,\, -2v,\, -2)\,.$$

Next, we find the normal vector:

$$\mathbf{x}_u \wedge \mathbf{x}_v = \left(-2u(1+r^2),\, 2v(1+r^2),\, 1-r^4\right), \quad \text{where } r^2 = u^2+v^2,$$

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = (1+r^2)^2.$$

Therefore,

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{1}{(1+u^2+v^2)}\left(-2u,\, 2v,\, 1-u^2-v^2\right).$$

The coefficients of the second fundamental form are given by the following inner products:

$$\begin{aligned} e = \langle N, \mathbf{x}_{uu}\rangle &= \frac{1}{(1+u^2+v^2)}\left(4u^2+4v^2+2(1-u^2-v^2)\right) = 2, \\ f = \langle N, \mathbf{x}_{uv}\rangle &= \frac{1}{(1+u^2+v^2)}(-4uv+4uv+0) = 0, \\ g = \langle N, \mathbf{x}_{vv}\rangle &= \frac{1}{(1+u^2+v^2)}\left(-4u^2-4v^2-2(1-u^2-v^2)\right) = -2. \end{aligned}$$

c. The shape operator in the (u,v) basis is given by $S = \mathrm{I}^{-1}\,\mathrm{II}$, where

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{pmatrix},$$

and

$$\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus,

$$S = \mathrm{I}^{-1}\,\mathrm{II} = \frac{1}{(1+u^2+v^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \frac{1}{(1+u^2+v^2)^2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The principal curvatures are the eigenvalues of the shape operator, which are easily seen to be

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

d. The lines of curvature correspond to the eigenvectors of the shape operator, which are ∂_u and ∂_v. Since the shape operator is diagonal in the $(\mathbf{x}_u, \mathbf{x}_v)$ basis, the lines of curvature are the coordinate curves $u = \text{const.}$ and $v = \text{const.}$.

e. For each p on an asymptotic curve, the normal curvature in the direction of the tangent vector is zero. The normal curvature k_n in the direction of a unit tangent vector $\mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v$ is given by

$$k_n = \langle S(\mathbf{t}), \mathbf{t}\rangle = \frac{2}{(1+u^2+v^2)^2}((\mathrm{d}u)^2 - (\mathrm{d}v)^2).$$

Setting $k_n = 0$ gives $(\mathrm{d}u)^2 = (\mathrm{d}v)^2$, which implies $\mathrm{d}u = \pm\mathrm{d}v$. Therefore, the asymptotic directions correspond to the curves where $u+v = \text{const.}$ and $u - v = \text{const.}$

From Do Carmo, the normal curvature is given by

$$k_n = k\langle n, N\rangle,$$

Exercise 2 (Do Carmo 3.3.8, Contact of Order ≥ 2 of Surfaces)**.** Two surfaces S and $\bar{S}$, with a common point p, have contact of order ≥ 2 at p if there exist parametrizations $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$ in p of S and $\bar{S}$, respectively, such that

$$\mathbf{x}_u = \bar{\mathbf{x}}_u, \quad \mathbf{x}_v = \bar{\mathbf{x}}_v, \quad \mathbf{x}_{uu} = \bar{\mathbf{x}}_{uu}, \quad \mathbf{x}_{uv} = \bar{\mathbf{x}}_{uv}, \quad \mathbf{x}_{vv} = \bar{\mathbf{x}}_{vv}.$$

a. Let S and $\bar{S}$ have contact of order ≥ 2 at p; $\mathbf{x}\colon U \to S$ and $\bar{\mathbf{x}}\colon U \to \bar{S}$ be arbitrary parametrizations in p of S and $\bar{S}$ respectively; and $f\colon V \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function in a neighborhood V of p in $\mathbb{R}^3$. Then the partial derivatives of order ≤ 2 of $f \circ \bar{\mathbf{x}}\colon U \to \mathbb{R}$ are zero in $\bar{\mathbf{x}}^{-1}(p)$ if and only if the partial derivatives of order ≤ 2 of $f \circ \mathbf{x}\colon U \to \mathbb{R}$ are zero in $\mathbf{x}^{-1}(p)$.

***b.** Let S and $\bar{S}$ have contact of order ≥ 2 at p. Let $z = f(x,y)$ and $z = \bar{f}(x,y)$ be the equations, in a neighborhood of p, of S and $\bar{S}$, respectively, where the xy-plane is the common tangent plane at $p = (0,0)$. Then the function $f(x,y) - \bar{f}(x,y)$ has all partial derivatives of order ≤ 2 equal to zero at $(0,0)$.

c. Let p be a point in a surface $S \subset \mathbb{R}^3$. Let $Oxyz$ be a Cartesian coordinate system for $\mathbb{R}^3$ such that $O = p$ and the xy-plane is the tangent plane of S at p. Show that the paraboloid

$$z = \tfrac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}),$$

obtained by neglecting third- and higher-order terms in the Taylor development around $p = (0,0)$, has contact of order ≥ 2 at p with S (the surface $(*)$ is called the <u>osculating paraboloid</u> of S at p).

***d.** If a paraboloid (the degenerate cases of plane and parabolic cylinder are included) has contact of order ≥ 2 with a surface S at p, then it is the osculating paraboloid of S at p.

***e.** If two surfaces have contact of order ≥ 2 at p, then the osculating paraboloids of S and $\bar{S}$ at p coincide. Conclude that the Gaussian and mean curvatures of S and $\bar{S}$ at p are equal.

***f.** The notion of contact of order ≥ 2 is invariant by diffeomorphisms of $\mathbb{R}^3$; that is, if S and $\bar{S}$ have contact of order ≥ 2 at p and $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism, then $\varphi(S)$ and $\varphi(\bar{S})$ have contact of order ≥ 2 at $\varphi(p)$.

***g.** If S and $\bar{S}$ have contact of order ≥ 2 at p, then

$$\lim_{r \to 0} \frac{d}{r^2} = 0,$$

where d is the length of the segment cut by the surfaces in a straight line normal to $T_p(S) = T_p(\bar{S})$, which is at a distance r from p.

Solution 2.

a. Suppose the partial derivatives of order ≤ 2 of $f \circ \bar{\mathbf{x}}$ are zero in $\bar{\mathbf{x}}^{-1}(p)$. Then, by the chain rule, we have

$$(f \circ \bar{\mathbf{x}})_u = \nabla f \cdot \bar{\mathbf{x}}_u = 0, \quad (f \circ \bar{\mathbf{x}})_v = \nabla f \cdot \bar{\mathbf{x}}_v = 0,$$
$$(f \circ \bar{\mathbf{x}})_{uu} = \nabla f \cdot \bar{\mathbf{x}}_{uu} + \bar{\mathbf{x}}_u^T H_f \bar{\mathbf{x}}_u = 0,$$
$$(f \circ \bar{\mathbf{x}})_{uv} = \nabla f \cdot \bar{\mathbf{x}}_{uv} + \bar{\mathbf{x}}_u^T H_f \bar{\mathbf{x}}_v = 0,$$
$$(f \circ \bar{\mathbf{x}})_{vv} = \nabla f \cdot \bar{\mathbf{x}}_{vv} + \bar{\mathbf{x}}_v^T H_f \bar{\mathbf{x}}_v = 0,$$

where H_f is the Hessian matrix of f at p. Since S and $\bar{S}$ have contact of order ≥ 2 at p, in the region $\mathbf{x}^{-1}(p)$ we have $(f \circ \mathbf{x})_{uu} = \nabla f \cdot \mathbf{x}_{uu} + \mathbf{x}_u^T H_f \mathbf{x}_u = \nabla f \cdot \bar{\mathbf{x}}_{uu} + \bar{\mathbf{x}}_u^T H_f \bar{\mathbf{x}}_u = 0$. Similarly, $(f \circ \mathbf{x})_{uv} = (f \circ \mathbf{x})_{vv} = (f \circ \mathbf{x})_u = (f \circ \mathbf{x})_v = 0$. The converse follows by symmetry.

b. Since S, $\overline{S}$ have $z = 0$ as the common tangent plane, their graph at $p = 0$ satisfy $f(0,0) = \overline{f}(0,0) = 0$ and $\nabla f(0,0) = \nabla \overline{f}(0,0) = 0$. Let's define the function $F : \mathbb{R}^3 \to \mathbb{R}$, such that $F(x,y,z) = z - \frac{1}{2} f_{xx}(0,0)x^2 - f_{xy}(0,0)xy - \frac{1}{2} f_{yy}(0,0)y^2$. Since F is a polynomial of x, y, x, it is differentiable. The parametrizations $\mathbf{x}$, $\overline{\mathbf{x}}$ for S and $\overline{S}$ at p are given by $\mathbf{x}(x,y) = (x, y, f(x,y))$ and $\overline{\mathbf{x}}(x,y) = \big(x, y, \overline{f}(x,y)\big)$, respectively. Then, $(F \circ \mathbf{x})(x,y) = f(x,y) - \frac{1}{2} f_{xx}(0,0)x^2 - f_{xy}(0,0)xy - \frac{1}{2} f_{yy}(0,0)y^2$, so all the partial derivatives of order ≤ 2

of $F \circ \mathbf{x}$ at $(0,0)$ are zero. By part **a.**, all the partial derivatives of order ≤ 2 of $F \circ \overline{\mathbf{x}}$ at $(0,0)$ are also zero. Therefore,

$$F \circ \overline{\mathbf{x}}(x,y) = \overline{f}(x,y) - \frac{1}{2} f_{xx}(0,0)x^2 - f_{xy}(0,0)xy - \frac{1}{2} f_{yy}(0,0)y^2$$

has all partial derivatives of order ≤ 2 vanish at p. Thus, the function $f(x,y) - \overline{f}(x,y)$ has all partial derivatives of order ≤ 2 vanish at p.

c. In a neighborhood of p, the surface S can be expressed as the graph of a function $z = f(x,y)$, where the xy-plane is the tangent plane at p. Since the xy-plane is the tangent plane at p, we have $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, so the Taylor expansion of $f(x,y)$ around p is given by

$$f(x,y) = \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) + R_3(x,y).$$

Let $\overline{S}$ be the paraboloid defined by

$$z = g(x,y) = \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right).$$

The parametrizations for S and $\overline{S}$ at p are given by $\mathbf{x}(x,y) = (x,y,f(x,y))$ and $\overline{\mathbf{x}}(x,y) = (x,y,g(x,y))$, respectively. The second-order partial derivatives of f and g at p are equal, since the remainder term $R_3(x,y)$ contains only terms of order ≥ 3. Therefore, by definition, S and $\overline{S}$ have contact of order ≥ 2 at p.

d. Suppose a paraboloid $\overline{S}$ has contact of order ≥ 2 with a surface S at p. Let the equation of S in a neighborhood of p be given by $z = f(x,y)$, where the xy-plane is the tangent plane at p. The equation of the paraboloid $\overline{S}$ can be expressed as

$$z = \overline{f}(x,y) = ax^2 + 2bxy + cy^2,$$

for some constants $a, b, c \in \mathbb{R}$. The second-order Taylor expansion of $f(x,y)$ around p is given by

$$f(x,y) = \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right).$$

Comparing this with the expression for $\overline{f}(x,y)$, we find that

$$a = \frac{1}{2} f_{xx}(0,0), \quad b = \frac{1}{2} f_{xy}(0,0), \quad c = \frac{1}{2} f_{yy}(0,0).$$

Thus, the paraboloid $\overline{S}$ is the osculating paraboloid of S at p as defined in **c.**.

e. Let P, $\overline{P}$ be the osculating paraboloids of S and $\overline{S}$, respectively. By **b.**, S, $\overline{S}$ have contact of order ≥ 2 at p with P, $\overline{P}$, respectively. Since S also has contact of order ≥ 2 with $\overline{S}$, all the partial derivatvies of order ≤ 2 of f and $\overline{f}$ vanish at p, where f, $\overline{f}$ are the equations in a neighborhood of p, of S and $\overline{S}$, respectively. Therefore,

$$\frac{1}{2}\left(f_{xx}(p)x^2 + 2f_{xy}(p)xy + f_{yy}(p)y^2\right) = \frac{1}{2}\left(\overline{f}_{xx}(p)x^2 + 2\overline{f}_{xy}(p)xy + \overline{f}_{yy}(p)y^2\right),$$

and the osculating paraboloids P and $\overline{P}$ coincide. Since the Gaussian and mean curvatures depend only on the partial derivatives of order ≤ 2 of the parametrization at p, the Gaussian and mean curvatures of S and $\overline{S}$ at p are equal.

f. Suppose S and $\overline{S}$ have contact of order ≥ 2 at p. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. The parametrizations for S and $\overline{S}$ at p are given by $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$, respectively. The parametrizations for $\varphi(S)$ and $\varphi(\overline{S})$ at $\varphi(p)$ are given by $\mathbf{y} = (\varphi \circ \mathbf{x})(u,v)$ and $\overline{\mathbf{y}} = (\varphi \circ \overline{\mathbf{x}})(u,v)$, respectively. Then, by the chain rule, we have

$$\mathbf{y}_u = \mathrm{d}\varphi_{\mathbf{x}} \cdot \mathbf{x}_u, \quad \mathbf{y}_v = \mathrm{d}\varphi_{\mathbf{x}} \cdot \mathbf{x}_v, \quad \mathbf{y}_{uu} = \mathrm{d}^2\varphi_{\mathbf{x}}(\mathbf{x}_u, \mathbf{x}_u) + \mathrm{d}\varphi_{\mathbf{x}} \cdot \mathbf{x}_{uu},$$

$$\mathbf{y}_{uv} = \mathrm{d}^2\varphi|_{\mathbf{x}}(\mathbf{x}_u, \mathbf{x}_v) + \mathrm{d}\varphi|_{\mathbf{x}} \cdot \mathbf{x}_{uv}, \quad \mathbf{y}_{vv} = \mathrm{d}^2\varphi|_{\mathbf{x}}(\mathbf{x}_v, \mathbf{x}_v) + \mathrm{d}\varphi|_{\mathbf{x}} \cdot \mathbf{x}_{vv},$$

and similarly for $\overline{\mathbf{y}}$, where $\mathrm{d}^2\phi|_{\mathbf{x}}$ is the bilinear differential of ϕ evaluated at $\mathbf{x}$.

Since S and $\overline{S}$ have contact of order ≥ 2 at p, it follows that $\mathbf{y}_u = \overline{\mathbf{y}}_u$, $\mathbf{y}_v = \overline{\mathbf{y}}_v$, $\mathbf{y}_{uu} = \overline{\mathbf{y}}_{uu}$, $\mathbf{y}_{uv} = \overline{\mathbf{y}}_{uv}$, and $\mathbf{y}_{vv} = \overline{\mathbf{y}}_{vv}$. Thus, $\varphi(S)$ and $\varphi(\overline{S})$ have contact of order ≥ 2 at $\varphi(p)$.

g. We may choose a Cartesian coordinate system $Oxyz$ such that $O = p$, and $z = 0$ is the common tangent plane of S and $\overline{S}$ at p. Let the equations of S and $\overline{S}$ in a neighborhood of p be given by $z = f(x, y)$ and $z = \overline{f}(x, y)$, respectively. Since S and $\overline{S}$ have contact of order ≥ 2 at p, by part **b.**, all the partial derivatives of order ≤ 2 of the function $G(x, y) \equiv f(x, y) - \overline{f}(x, y)$ vanish at p. Therefore, $G(0, 0) = \nabla G(0, 0) = \nabla^2 G(0, 0) = 0$, where $\nabla^2 G$ is the Hessian matrix of G. Take a point $q = (x, y, 0) \in T_p(S)$ in the tangent plane, a distance $r = \sqrt{x^2 + y^2}$ from p. The straight line L_q normal to the tangent plane passing through q intersects the surfaces S and $\overline{S}$ at the points $(x, y, f(x, y))$ and $(x, y, \overline{f}(x, y))$, respectively, and $d = |f(x, y) - \overline{f}(x, y)| = |G(x, y)|$.

Define the function $g(t) = G(tu)$ for a fixed u, where $u \in \mathbb{R}^2$ is a unit vector such that $(x, y) = ru$. Then g is differentiable, and $g(0) = g'(0) = g''(0) = 0$, since all the partial derivatives of order ≤ 2 of F vanish at p. By Taylor's formula with remainder, we have

$$g(t) = g(0) + g'(0) + \int_0^t \mathrm{d}s\,(t - s)g''(s) = \int_0^t \mathrm{d}s\,(t - s)g''(s)$$

for all t in a neighborhood of 0. Next we will bound $|g|$. Since F is smooth, $\nabla^2 F$ is continuous, so for all $\varepsilon > 0$ there exists $\delta > 0$, such that $\|(x, y)\| < \delta$ implies $\|\nabla^2 F(x, y)\| < 2\varepsilon$. Hence, for $t < \delta$, $|g''(t)| = |u^T \nabla^2 F u| \leq |\nabla^2 F| \|u^2\| < 2\varepsilon$. Take $t = r < \delta$, then we have

$$\begin{aligned} |G(ru)| = |g(r)| &= \left| \int_0^r \mathrm{d}s\,(r - s)g''(s) \right| \leq \int_0^r \mathrm{d}s\,(r - s)|g''(s)| \\ &\leq \int_0^r \mathrm{d}s\,(r - s)2\varepsilon r^2 = \varepsilon r^2. \end{aligned}$$

Notice that $d = G(x, y) = G(ru)$, so for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\frac{d}{r^2} < \varepsilon$ whenever $\sqrt{x^2 + y^2} < \delta$. This proves the desired result.

Exercise 3 (Do Carmo 3.3.13)**.** Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map (a similarity) defined by $F(p) = cp$, $p \in \mathbb{R}^3$, c a positive constant. Let $S \subset \mathbb{R}^3$ be a regular surface and set $\bar{S} = F(S)$. Show that $\bar{S}$ is a regular surface, and find formulas relating the Gaussian and mean curvatures, K and H, of S with the Gaussian and mean curvatures, $\bar{K}$ and $\bar{H}$, of $\bar{S}$.

Solution 3.

1. Let $\mathbf{x} : U \subseteq \mathbb{R} \to S$ be a local parametrization of S. Let $\overline{S} = F(S)$, then $\overline{\mathbf{x}} = F \circ \mathbf{x} : U \to \overline{S}$ is a local parametrization of $\overline{S}$. The map F is smooth, and since $\mathrm{d}F = c\,\mathrm{Id}$ is an isomorphism, $\mathrm{d}\overline{\mathbf{x}} = \mathrm{d}F \circ \mathrm{d}\mathbf{x} = c\mathrm{d}\mathbf{x}$ has rank 2 whenever $\mathrm{d}\mathbf{x}$ has rank 2. Thus, $\overline{\mathbf{x}}$ is a homeomorphism onto its image and $\mathrm{d}\overline{x}$ is injective (hence an immersion). Therefore, $\overline{S}$ is a regular surface.

2. For any local parametrization $\mathbf{x}$ and $\overline{\mathbf{x}}$, we have $\overline{\mathbf{x}} = c\mathbf{x}$. Thus,

$$\overline{\mathbf{x}}_u = c\mathbf{x}_u, \quad \overline{\mathbf{x}}_v = c\mathbf{x}_v, \quad \overline{\mathbf{x}} \wedge \overline{\mathbf{x}}_v = c^2(\mathbf{x}_u \wedge \mathbf{x}_v).$$

Hence, the normal for $\overline{S}$ satisfies $\overline{N} = N$. Write the Weingarten map for S and $\overline{S}$ as $\mathcal{S}$ and $\overline{\mathcal{S}}$, respectively. By definition, $\mathrm{d}N = -\mathcal{S} \circ \mathrm{d}\mathbf{x}$, so

$$\mathrm{d}\overline{N} = \mathrm{d}N = -\mathcal{S} \circ \mathrm{d}\mathbf{x} = -\mathcal{S} \circ \frac{1}{c}\,\mathrm{d}\overline{\mathbf{x}} = -\left(\frac{1}{c}\mathcal{S}\right) \circ \mathrm{d}\overline{\mathbf{x}}.$$

Therefore, $\overline{\mathcal{S}} = \frac{1}{c}\mathcal{S}$, and the principle curvatures satisfy $\overline{k}_i = \frac{1}{c}k_i$, since they are the eigenvalues of $\mathcal{S}$. The Gaussian curvature K and mean curvature H of S are then given by

$$\overline{K} = \overline{k}_1\overline{k}_2 = \frac{1}{c^2}k_1k_2 = \frac{1}{c^2}K,$$
$$\overline{H} = \frac{\overline{k}_1 + \overline{k}_2}{2} = \frac{1}{c}\frac{k_1+k_2}{2} = \frac{1}{c}H.$$

Exercise 4 (Do Carmo 3.3.24, Local Convexity and Curvature)**.**

A surface $S \subset \mathbb{R}^3$ is locally convex at a point $p \in S$ if there exists a neighborhood $V \subset S$ of p such that V is contained in one of the closed half-spaces determined by $T_p(S)$ in $\mathbb{R}^3$. If, in addition, V has only one common point with $T_p(S)$, then S is called strictly locally convex at p.

a. Prove that S is strictly locally convex at p if the principal curvatures of S at p are nonzero with the same sign (that is, the Gaussian curvature $K(p)$ satisfies $K(p) > 0$).

b. Prove that if S is locally convex at p, then the principal curvatures at p do not have different signs (thus, $K(p) \geq 0$).

c. To show that $K \geq 0$ does not imply local convexity, consider the surface

$$f(x,y) = x^3(1+y^2),$$

defined in the open set $U = \{(x,y) \in \mathbb{R}^2 : y^2 < \frac{1}{2}\}$. Show that the Gaussian curvature of this surface is nonnegative on U and yet the surface is not locally convex at $(0,0) \in U$ (a deep theorem, due to R. Sacksteder, implies that such an example cannot be extended to the entire $\mathbb{R}^2$ if we insist on keeping the curvature nonnegative; cf. Remark 3 of Sec. 5-6).

***d.** The example of part (c) is also very special in the following local sense. Let p be a point in a surface S, and assume that there exists a neighborhood $V \subset S$ of p such that the principal curvatures on V do not have different signs (this does not happen in the example of part c). Prove that S is locally convex at p.

Solution 4.

a. Without loss of generality, assume $k_1, k_2 > 0$, since if both are negative, just replace the chosen unit normal by its negative. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ be a local parametrization of S such that $\{\mathbf{x}_u, \mathbf{x}_v\}$ is an orthonormal basis of principle directions at $p \in S$, where $p = \mathbf{x}(0,0)$. Following the definition of Exercise 3.3.22, define the height function $h : U \to \mathbb{R}$ of S relative to $T_p(S)$ by

$$h(u,v) = \langle \mathbf{x}(u,v) - p, N(p) \rangle,$$

where $N(p)$ is the unit normal vector p. We compute the derivatives as follows:

$$\begin{aligned}
h(p) &= \langle \mathbf{x}(0,0) - p, N(p) \rangle = 0,\\
h_u(p) &= \langle \mathbf{x}_u(0,0), N(p) \rangle = 0,\\
h_v(p) &= \langle \mathbf{x}_v(0,0), N(p) \rangle = 0,\\
h_{uu}(p) &= \langle \mathbf{x}_{uu}(0,0), N(p) \rangle = e(p),\\
h_{uv}(p) &= \langle \mathbf{x}_{uv}(0,0), N(p) \rangle = f(p),\\
h_{vv}(p) &= \langle \mathbf{x}_{vv}(0,0), N(p) \rangle = g(p),
\end{aligned}$$

where $h_{ij}(p)$ are the coefficients of the second fundamental form at p. Since $\mathbf{x}_u(0,0)$ and $\mathbf{x}_v(0,0)$ are principle directions and orthonormal, we have $e(p) = k_1$, $f(p) = 0$, and $g(p) = k_2$. Thus, the Hessian matrix of h at p is given by

$$\nabla^2 h(p) = \begin{pmatrix} h_{uu}(p) & h_{uv}(p) \\ h_{uv}(p) & h_{vv}(p) \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},$$

and Taylor expansion gives

$$h(u, v) = \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + o\left(u^2 + v^2\right),$$

Since $k_1, k_2 > 0$, the quadratic form $Q = \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right)$ associated with $\nabla^2 h(p)$ is positive definite. Hence, there exists a neighborhood $W \subset U$ of p and some $c > 0$ such that $Q(u, v) > c(u^2 + v^2)$ for all $(u, v) \in W$. Now since

$$\frac{h(u, v) - Q(u, v)}{u^2 + v^2} \to 0 \quad \text{as } (u, v) \to (0, 0),$$

there exists a radius $\delta > 0$ such that $\sqrt{u^2 + v^2} < \delta$ implies $|h(u, v) - Q(u, v)| < \frac{c}{2}(u^2 + v^2)$. Therefore, for all $(u, v) \in W$ with $\sqrt{u^2 + v^2} < \delta$, we have

$$h(u, v) \geq Q(u, v) - |h(u, v) - Q(u, v)| > c(u^2 + v^2) - \frac{c}{2}(u^2 + v^2) = \frac{c}{2}(u^2 + v^2) > 0,$$

with $h(u, v) = 0$ if and only if $(u, v) = (0, 0)$. Thus, the neighborhood $V = \mathbf{x}(W \cap \{(u, v) : \sqrt{u^2 + v^2} < \delta\})$ of p is contained in the half-space $H^+ = \{q \in \mathbb{R}^3 \mid \langle q - p, N(p)\rangle \geq 0\}$, and V has only one common point with $T_p(S)$. Therefore, S is strictly locally convex at p.

b. Suppose S is locally convex at p, so there exists a neighborhood $V \subset S$ of p such that V is contained in one of the closed half-spaces determined by $T_p(S)$. Define the height function as above, by local convexity we may choose an orientation $N(p)$ such that $h(u, v) \geq 0$ in a neighborhood of $(0, 0)$, and $h(0, 0) = h_u(0, 0) = h_v(0, 0) = 0$. Suppose that the principal curvatures at p have different signs, say $k_1 > 0 > k_2$. Then, along the coordinate axes, we have $h(u, 0) = \frac{1}{2}k_1 u^2 > 0$ for all $|u| < \delta_u$, and $h(0, v) = \frac{1}{2}k_2 v^2 < 0$ for all $|v| < \delta_v$. Hence, in every neighborhood of $(0, 0)$, we can find points such that $h(u, v) > 0$ and others such that $h(u, v) < 0$, contradicting local convexity. Therefore, the principal curvatures at p do not have different signs, and hence $K(p) \geq 0$.

c. The Gaussian curvature K of the surface defined by $z = f(x, y)$ is given by

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

Let's compute the necessary partial derivatives of $f(x, y) = x^3(1 + y^2)$:

$$f_x = 3x^2(1 + y^2), \quad f_y = 2x^3 y, \quad f_{xx} = 6x(1 + y^2), \quad f_{yy} = 2x^3, \quad f_{xy} = 6x^2 y.$$

Then, we have

$$K = \frac{(6x(1 + y^2))(2x^3) - (6x^2 y)^2}{(1 + (3x^2(1 + y^2))^2 + (2x^3 y)^2)^2} = \frac{12x^4(1 - 2y^2)}{(1 + 9x^4(1 + y^2)^2 + 4x^6 y^2)^2} \geq 0.$$

However, the surface is not locally convex at $(0, 0)$, since for any neighborhood V of $(0, 0)$, there exist points with both positive and negative x values, and hence z-coordinates, so V is not contained in one of the closed half-spaces determined by the tangent plane at $(0, 0)$.

d. Suppose $V \subseteq S$ is a neighborhood of p such that the principal curvatures on V do not have different signs. Without loss of generality, assume $k_1(q), k_2(q) \geq 0$ for all $q \in V$, since if at some point one of them were positive and later negative, it would have to cross zero alone, producing a point where the two have different signs, which is excluded by definition of V. Follow the steps of **a.**, we define the height function $h : U \to \mathbb{R}$ of S relative to $T_p(S)$ by $h(u, v) = \langle \mathbf{x}(u, v) - p, N(p)\rangle$. Pick an orthonormal basis of principal directions $\{\mathbf{x}_u, \mathbf{x}_v\}$. The Hessian matrix of h at p is given, again, by

$$\nabla^2 h(p) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Near $(0, 0)$, we have

$$h(u, v) = \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + o\left(u^2 + v^2\right),$$

and the quadratic form $Q = \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right)$ is positive-definite. Now we consider two cases:

(a) At least one of the principal curvatures at p is positive, say $k_1 > 0$. Then, there exists a neighborhood $W \subset U$ of p and some $c > 0$ such that $Q(u, v) > c(u^2 + v^2)$ for all $(u, v) \in W$. Following the same steps as in **a.**, we can show local convexity at p.

(b) Both principal curvatures at p are zero, i.e., $k_1 = k_2 = 0$, so $Q = 0$. Since the principal curvatures are continuous functions on S, we have $h(0, 0) = 0$ and $h(u, v) \geq 0$ in a neighborhood of p. Therefore, S is locally convex at p.