2025 Fall Introduction to Geometry

Homework 9 (Due Nov 21, 2025)

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Exercise 1 (Do Carmo 3.4.2)**.** Prove that the vector field obtained on the torus by parametrizing all its meridians by arc length and taking their tangent vectors (Example 1) is differentiable.

Solution 1. From Do Carmo 3.4 Definition 1, a vector field w is differentiable if, for some parametrization $\mathbf{x} : U \to \mathbb{R}^3$, the functions $a(u,v)$ and $b(u,v)$ given by $w = a(u,v)\mathbf{x}_u + b(u,v)\mathbf{x}_v$ are differentiable on U. Parametrize the torus by

$$\mathbf{x}(u,v) = \left((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v\right),$$

where R is the distance from the center of the tube to the center of the torus, and r is the radius of the tube. Fix $\theta = \theta_0$ and vary $\phi = \frac{s}{r}$, we have

$$\alpha_{\theta_0}(s) = \mathbf{x}(\theta_0, s/r) = \left((R + r\cos s/r)\cos\theta_0, (R + r\cos s/r)\sin\theta_0, r\sin s/r\right).$$

Then the vector field obtained by parametrizing the meridians by arc length is given by

$$w(\mathbf{x}(\theta_0, s/r)) = \alpha'_{\theta_0}(s) = \left(-\sin s/r\cos\theta_0, -\sin s/r\sin\theta_0, \cos s/r\right).$$

Let $w(\mathbf{x}(\theta,\phi)) = a(\theta,\phi)\mathbf{x}_\theta + b(\theta,\phi)\mathbf{x}_\phi$, we have

$$\mathbf{x}_\theta = \left(-(R + r\cos\phi)\sin\theta, (R + r\cos\phi)\cos\theta, 0\right),$$

$$\mathbf{x}_\phi = \left(-r\sin\phi\cos\theta, -r\sin\phi\sin\theta, r\cos\phi\right).$$

Comparing the coefficients, we get $a(\theta,\phi) = 0$, $b(\theta,\phi) = \frac{1}{r}$. Since they are both differentiable, w is differentiable.

Exercise 2 (Do Carmo 3.4.3)**.** Prove that a vector field w defined on a regular surface $S \subset \mathbb{R}^3$ is differentiable if and only if it is differentiable as a map $w : S \to \mathbb{R}^3$.

Solution 2. Suppose w is differentiable as a vector field. Then, there exist a parametrization $\mathbf{x} : U \to S$ such that $w = a(u,v)\mathbf{x}_u + b(u,v)\mathbf{x}_v$ for differentiable functions $a(u,v)$ and $b(u,v)$. Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are differentiable, $w \circ \mathbf{x} = a(u,v)\mathbf{x}_u + b(u,v)\mathbf{x}_v$ is differentiable. Thus, w is differentiable as a map. Conversely, suppose w is differentiable as a map $w : S \to \mathbb{R}^3$. Then, for any parametrization $\mathbf{x} : U \to S$ and each $(u,v) \in U$, since $\{\mathbf{x}_u, \mathbf{x}_v\}$ forms a basis for $T_p(S)$, there exist scalars $a(u,v)$ and $b(u,v)$ such that $(w \circ \mathbf{x})(u,v) = a(u,v)\mathbf{x}_u + b(u,v)\mathbf{x}_v$. Then, we have

$$\langle w, \mathbf{x}_u\rangle = a\langle \mathbf{x}_u, \mathbf{x}_u\rangle + b\langle \mathbf{x}_v, \mathbf{x}_u\rangle, \quad \langle w, \mathbf{x}_v\rangle = a\langle \mathbf{x}_u, \mathbf{x}_v\rangle + b\langle \mathbf{x}_v, \mathbf{x}_v\rangle.$$

Let $\alpha = \langle w, \mathbf{x}_u\rangle$, $\beta = \langle w, \mathbf{x}_v\rangle$, then

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}$$

Since $\{\mathbf{x}_u, \mathbf{x}_v\}$ are linearly independent, $\det(\mathrm{I}) = EG - F^2 \neq 0$, and we have

$$a = \frac{G\alpha - F\beta}{EG - F^2}, \quad b = \frac{-F\alpha + E\beta}{EG - F^2}.$$

Since w, $\mathbf{x}_u$ and $\mathbf{x}_v$ are differentiable, α and β are differentiable. Also, since E, F and G are differentiable, $a(u,v)$ and $b(u,v)$ are differentiable. Therefore, w is differentiable as a vector field.

Exercise 3 (Do Carmo 3.4.6)**.** A straight line r meets the z axis and moves in such a way that it makes a constant angle $\alpha \neq 0$ with the z axis and each of its points describes a helix of pitch $c \neq 0$ about the z axis. The figure described by r is the trace of the parametrized surface (see Fig. 3–32)

$$x(u, v) = (v \sin\alpha \cos u,\ v \sin\alpha \sin u,\ v \cos\alpha + cu).$$

The map x is easily seen to be a regular parametrized surface. Restrict the parameters (u, v) to an open set U so that $x(U) = S$ is a regular surface.

a. Find the orthogonal family (cf. Example 3) to the family of coordinate curves $u = \text{const.}$

b. Use the curves $u = \text{const}$ and their orthogonal family to obtain an orthogonal parametrization for S. Show that in the new parameters $(\tilde{u}, \tilde{v})$ the coefficients of the first fundamental form are

$$\tilde{G} = 1, \qquad \tilde{F} = 0, \qquad \tilde{E} = \{c^2 + (\tilde{v} - c\tilde{u}\cos\alpha)^2\} \sin^2\alpha.$$

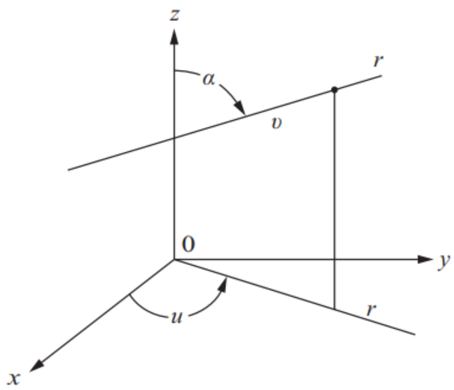

Figure 3-32

Solution 3.

a. The coordinate curves $u = \text{const}$ have tangent vectors $\mathbf{x}_v$. Let the curve be given by $v = v(t)$, $u = u_0$. Then, its tangent vector is $\mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)$. Orthogonaity gives $\langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_v \rangle = 0$, and hence $Fu' + Gv' = 0$. Let's calculate the coefficients of the first fundamental form:

$$\mathbf{x}_u = (-v \sin\alpha \sin u, v \sin\alpha \cos u, c), \quad \mathbf{x}_v = (\sin\alpha \cos u, \sin\alpha \sin u, \cos\alpha).$$

Thus, we have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = v^2 \sin^2\alpha + c^2, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = c\cos\alpha, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1.$$

Treating $v(t)$ as a function of u, i.e. $v(t) = v(t(u))$, we have

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{F}{G} = -c\cos\alpha \implies v(u) = -cu\cos\alpha + k.$$

Thus, the orthogonal family to the curves $u = \text{const}$ is given by $cu\cos\alpha + v = k$ in the (u, v)-plane.

b. We have two transverse families of curves in the (u, v)-plane, given by $u = \text{const.}$ and $cu\cos\alpha + v = \text{const.}$. Let's define new parameters $(\tilde{u}, \tilde{v})$ by

$$\tilde{u} = u, \quad \tilde{v} = cu\cos\alpha + v.$$

The parametrization in the new parameters is given by $\tilde{\mathbf{x}}(\tilde{u}, \tilde{v}) = \mathbf{x}(u, v) = \mathbf{x}(\tilde{u}, \tilde{v} - c\tilde{u}\cos\alpha)$. Let's calculate the coefficients of the first fundamental form $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ in the new parameters:

$$\tilde{\mathbf{x}}_{\tilde{u}} = \mathbf{x}_u u_{\tilde{u}} + \mathbf{x}_v v_{\tilde{u}} = \mathbf{x}_u - c\cos\alpha \mathbf{x}_v,$$

$$\tilde{\mathbf{x}}_{\tilde{v}} = \mathbf{x}_u u_{\tilde{v}} + \mathbf{x}_v v_{\tilde{v}} = \mathbf{x}_v.$$

Substituting in the values of E, F, and G calculated in part **a.**, we have

$$\begin{aligned}
\tilde{E} &= \langle \tilde{\mathbf{x}}_{\tilde{u}}, \tilde{\mathbf{x}}_{\tilde{u}} \rangle = \langle \mathbf{x}_u - c\cos\alpha \mathbf{x}_v, \mathbf{x}_u - c\cos\alpha \mathbf{x}_v \rangle \\
&= E - 2c\cos\alpha F + c^2\cos^2\alpha G, \\
&= (v^2\sin^2\alpha + c^2) - 2c^2\cos^2\alpha + c^2\cos^2\alpha = (v^2 + c^2\sin^2\alpha)\sin^2\alpha \\
&= \{c^2 + (\tilde{v} - c\tilde{u}\cos\alpha)^2\}\sin^2\alpha. \\
\tilde{F} &= \langle \tilde{\mathbf{x}}_{\tilde{u}}, \tilde{\mathbf{x}}_{\tilde{v}} \rangle = \langle \mathbf{x}_u - c\cos\alpha \mathbf{x}_v, \mathbf{x}_v \rangle = F - c\cos\alpha G = 0, \\
\tilde{G} &= \langle \tilde{\mathbf{x}}_{\tilde{v}}, \tilde{\mathbf{x}}_{\tilde{v}} \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = G = 1.
\end{aligned}$$

Exercise 4 (Do Carmo 3.4.7)**.** Define the derivative $w(f)$ of a differentiable function $f : U \subset S \to \mathbb{R}$ relative to a vector field w in U by

$$w(f)(q) = \frac{d}{dt}(f \circ \alpha)\bigg|_{t=0}, \qquad q \in U,$$

where $\alpha : I \to S$ is a curve such that $\alpha(0) = q$ and $\alpha'(0) = w(q)$.

Prove that:

a. w is differentiable in U if and only if $w(f)$ is differentiable for all differentiable f in U.

b. Let λ, μ be real numbers and $g : U \subset S \to \mathbb{R}$ be a differentiable function on U; then

$$w(\lambda f + \mu f') = \lambda w(f) + \mu w(f'), \qquad w(fg) = w(f)g + f w(g).$$

Solution 4.

a. Suppose w is differentiable in U, then it is differentiable as a map $w : U \to \mathbb{R}^3$ by Exercise 3.4.3. For any differentiable function $f : U \to \mathbb{R}$, let $\mathbf{x} : V \to U$ be a local parametrization of U, and (u, v) a local coordinate. Then, we have

$$(w \circ \mathbf{x})(u, v) = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v,$$

where a, b are differentiable functions. Fix $q = \mathbf{x}(u, v) \in U$ and a curve $\alpha = \mathbf{x}(u(t), v(t))$ such that $\alpha(0) = q$, $\alpha'(0) = w(q)$. Let $\phi(u, v) = (f \circ \mathbf{x})(u, v)$, then, we have

$$w(f)(q) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha)(0) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t), v(t))\bigg|_{t=0} = \phi_u u'(0) + \phi_v v'(0),$$

and notice that in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, $(u'(t), v'(t)) = (a(u, v), b(u, v))$, so

$$w(f)(q) = \phi_u u'(0) + \phi_v v'(0) = \phi_u a(u, v) + \phi_v b(u, v)$$

is differentiable as a function of (u, v). Since $\mathbf{x}$ is a local parametrization, $w(f)$ is differentiable in U. Conversely, let π_i be the standard projection, we have $f_i = \pi_i|_U : U \to \mathbb{R}$. By hypothesis, each $w(f_i)$ is differentiable. Fix $q \in U$ and a curve α such that $\alpha(0) = q$, $\alpha'(0) = w(q)$. Then

$$w(f_i)(q) = \frac{\mathrm{d}}{\mathrm{d}t}(f_i \circ \alpha)(0) = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_i \circ \alpha)(0) = (w(q))_i\,,$$

and

$$w(q) = (w(f_1)(q), w(f_2)(q), w(f_3)(q))\,.$$

Since each component is differentiable, w is differentiable as a map $w : U \to \mathbb{R}^3$, and hence differentiable as a vector field in U by Exercise 3.4.3.

b. Let $q \in U$, $\alpha : I \to S$ be a curve such that $\alpha(0) = q$ and $\alpha'(0) = w(q)$. Then, we have

$$\begin{aligned} w(\lambda f + \mu f') &= \frac{\mathrm{d}}{\mathrm{d}t} \left((\lambda f + \mu f') \circ \alpha)\right|_{t=0} \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}t} \left(f \circ \alpha)\right|_{t=0} + \mu \frac{\mathrm{d}}{\mathrm{d}t} \left(f' \circ \alpha)\right|_{t=0} \\ &= \lambda w(f) + \mu w(f'), \end{aligned}$$

and

$$\begin{aligned} w(fg) &= \frac{\mathrm{d}}{\mathrm{d}t} \left((fg) \circ \alpha)\right|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left((f \circ \alpha)(g \circ \alpha))\right|_{t=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha)\right|_{t=0} (g \circ \alpha)(0) + (f \circ \alpha)(0) \left.\frac{\mathrm{d}}{\mathrm{d}t}(g \circ \alpha)\right|_{t=0} \\ &= w(f)g(q) + f(q)w(g). \end{aligned}$$

Exercise 5 (Do Carmo 3.4.8)**.** Show that if w is a differentiable vector field on a surface S and $w(p) \neq 0$ for some $p \in S$, then it is possible to parametrize a neighborhood of p by $x(u, v)$ in such a way that $x_u = w$.

Solution 5. Let's express w in a local parametrization $\mathbf{x} : U \to S$ in a neighborhood of $p = \mathbf{x}(0, 0)$. Let (u, v) be a local coordinate, then, by a slight abuse of notation,

$$w(u, v) \equiv (w \circ \mathbf{x})(u, v) = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v,$$

where $a(u, v)$, $b(u, v)$ are differentiable functions.

Claim. Let $\mathbf{a}(u, v) = (a(u, v), b(u, v))$. Suppose $\mathrm{d}\mathbf{a} \neq 0$, then there exists a neighborhood V of p and coordinates $(\tilde{u}, \tilde{v})$ such that $\mathbf{a} = a(\tilde{u}, \tilde{v})$. I.e. $w = (1, 0)$ in the basis $\{\tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v\} = \{\mathbf{x}_{\tilde{u}}, \mathbf{x}_{\tilde{v}}\}$.

Proof. Let (u, v) be a local coordinate in a neighborhood of p. Since $\mathrm{d}\mathbf{a} = \mathbf{a}_u \mathrm{d}u + \mathbf{a}_v \mathrm{d}v$ and $\mathrm{d}\mathbf{a}_p \neq 0$, at least one of $\mathbf{a}_u(p)$ and $\mathbf{a}_v(p)$ is non-zero. Without loss of generality, suppose $\mathbf{a}_u(p) \neq 0$. Then, by the Inverse Function Theorem, there exists a neighborhood V of p such that the map $\psi : V \to \mathbb{R}^2$ defined by $\psi(u, v) = (a(u, v), v)$ is a diffeomorphism onto its image. Let $(\tilde{u}, \tilde{v}) = \psi(u, v)$, then we have $\mathbf{a} = a(\tilde{u}, \tilde{v})$, as desired. □

Let $\Phi(t, \mathbf{x}(0, 0))$ be the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathbf{a}(y), \quad y(0) = \mathbf{x}(0, 0),$$

and let $\phi(u, v) = \Phi(u, (0, v))$. By the smooth dependence of solution of an ODE on initial conditions, Φ, and hence ϕ, is differentiable. Then, we have

$$\frac{\partial}{\partial u}\phi(u, v) = \mathbf{a}(\phi(u, v)) = w(\phi(u, v)).$$

Furthermore, since $\phi(0, v) = \Phi(0, (0, v)) = (0, v)$, we have $\mathrm{d}\phi_p = 1$, and hence ϕ is a local parametrization around p. Let $\tilde{\mathbf{x}}(u, v) = \phi(u, v)$, then we have $\tilde{\mathbf{x}}_u = w(\tilde{\mathbf{x}}(u, v))$.