2025 Fall Introduction to Geometry

Solutions to Exercises in Do Carmo

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1 Chapter 2

1.1 Chapter 2.1

1.2 Chapter 2.2

Definition 1 (regular surface)**.** A subset $S \subseteq \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood $V \subseteq \mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ such that

(i) $\mathbf{x}$ is (infinitely) differentiable.

(ii) $\mathbf{x}$ is a homeomorphism, i.e. $\mathbf{x}$ is a bijection, and both $\mathbf{x}$ and $\mathbf{x}^{-1}$ are continuous.

(iii) For each $q \in U$, the differential $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (the regularity condition).

The mapping $\mathbf{x}$ is called a parametrization of the surface S or a system of local coordinates around the point p. The neighborhood $V \cap S$ of p is called a coordinate neighborhood.

Definition 2 (regular and critical value)**.** Given a differential map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined on an open set U, we say that $p \in U$ is a critical value of F is the differential $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. Otherwise, p is called a regular value of F.

Proposition 1. If $f : U \to R$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.

Proposition 2. If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Proposition 3. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$.

Proposition 4. Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subseteq S$ such that the conditions 1 and 3 of Def. 1 (for a regular surface) hold. Assume that $\mathbf{x}$ is one-to-one. Then $\mathbf{x}^{-1}$ is continuous.

Exercise 1. Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.

Solution 1.

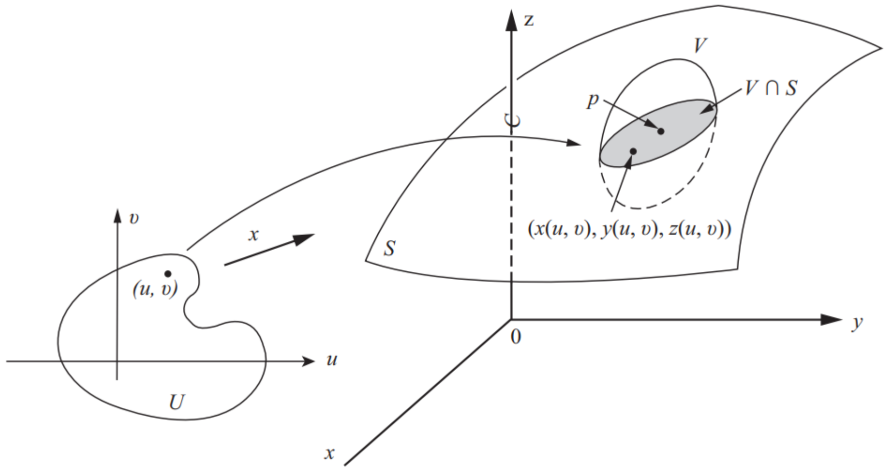

Figure 2-1

Exercise 2. Is the set $\{(x, y, z) \in \mathbb{R}^3,\ z = 0 \text{ and } x^2 + y^2 \leq 1\}$ a regular surface? Is the set $\{(x, y, z) \in \mathbb{R}^3,\ z = 0 \text{ and } x^2 + y^2 < 1\}$ a regular surface?

Solution 2.

Exercise 3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Solution 3.

Exercise 4. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet that $f^{-1}(0)$ is a regular surface.

Solution 4.

Exercise 5 (*). Let $P = \{(x, y, z) \in \mathbb{R}^3,\ x = y\}$ (a plane) and let $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$x(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in \mathbb{R}^2;\ u > v\}$. Clearly, $x(U) \subset P$. Is x a parametrization of P?

Solution 5.

Exercise 6. Give another proof of Proposition 1 by applying Proposition 2 to $h(x, y, z) = f(x, y) - z$.

Solution 6.

Exercise 7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

Solution 7.

Exercise 8. Let $x(u, v)$ be as in Definition 1. Verify that $dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one if and only if

$$\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} \neq 0.$$

Solution 8.

Exercise 9. Let V be an open set in the xy-plane. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3;\ z = 0 \text{ and } (x, y) \in V\}$$

is a regular surface.

Solution 9.

Exercise 10. Let C be a figure "8" in the xy-plane and let S be the cylindrical surface over C (Fig. 2-11); that is,

$$S = \{(x, y, z) \in \mathbb{R}^3;\ (x, y) \in C\}.$$

Is the set S a regular surface?

Solution 10.

Exercise 11. Show that the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \ ;\ z = x^2 - y^2\}$$

is a regular surface and check that parts (a) and (b) are parametrizations for S:

(a) $\mathbf{x}(u, v) = (u + v,\ u - v,\ 4uv), \quad (u, v) \in \mathbb{R}^2$.

(b) $\mathbf{x}(u, v) = (u \cosh v,\ u \sinh v,\ u^2), \quad (u, v) \in \mathbb{R}^2,\ u \neq 0$.

Which parts of S do these parametrizations cover?

Solution 11.

Notice that $z(x, y) = x^2 - y^2$ is a differentiable function from the open set $U = \mathbb{R}^2$ to $\mathbb{R}$, so by Proposition 2.2.1 in Do Carmo, S is a regular surface. Recall that a map $\mathbf{x} : U \to V \cap S$ if $\mathbf{x}$ is differentiable, a homeomorphism, and $\mathrm{d}\mathbf{x}_p$ is one-to-one for all $p \in U$.

(a) The map $\mathbf{x}$ is a polyminial in u and v, so it is differentiable. By explicit calculation,

$$\mathrm{d}\mathbf{x}_q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{pmatrix}$$

in the canonical basis, so $|\partial(x,y)/\partial(u,v)| = 2$ and $d\mathbf{x}$ is one-to-one. To show that $\mathbf{x}$ is a homeomorphism, observe that for any $(x,y,z) \in S$, we have $z = x^2 - y^2$, so $z = (u+v)^2 - (u-v)^2 = 4uv$, and

$$u = \frac{x+y}{2}, \quad v = \frac{x-y}{2}$$

from the remaining equations. This determines a unique (u,v) for each $(x,y,z) \in S$, and we can conclude that the inverse map $\mathbf{x}^{-1}$ exists and is continuous.

(b) The map $\mathbf{x}$ is a composition of polynomials and exponential functions, so it is differentiable. By explicit calculation,

$$d\mathbf{x}_q = \begin{pmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \\ 2u & 0 \end{pmatrix}$$

in the canonical basis, so $|\partial(x,y)/\partial(u,v)| = u$, and $d\mathbf{x}$ is one-to-one for $u \neq 0$. To show that $\mathbf{x}$ is a homeomorphism, observe that for any $(x,y,z) \in S$ with $x^2 - y^2 > 0$, we have $z = x^2 - y^2$, so $z = u^2(\cosh^2 v - \sinh^2 v) = u^2$, and

$$u = \pm\sqrt{x^2 - y^2}, \quad v = \tanh^{-1}\frac{y}{x}$$

from the remaining equations. This determines a unique (u,v) for each $(x,y,z) \in S$ with $x^2 - y^2 > 0$, and we can conclude that the inverse map $\mathbf{x}^{-1}$ exists and is continuous.

Parametrization (a) covers the whole surface S, while parametrization (b) only covers the parts of S where $|x| > |y|$.

Remark. The graph of $z = f(x,y) = x^2 - y^2$ is a hyperbolic paraboloid, also known as saddle.

Exercise 12. Show that $x : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$x(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u), \quad a, b, c \neq 0, \quad 0 < u < \pi,\ 0 < v < 2\pi,$$

is a parametrization for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Describe geometrically the curves $u = \text{const.}$ on the ellipsoid.

Solution 12.

Exercise 13 (*). Find a parametrization for the hyperboloid of two sheets

$$\{(x,y,z) \in \mathbb{R}^3 \ : \ -x^2 - y^2 + z^2 = 1\}.$$

Solution 13.

Exercise 14. A half-line $[0,\infty)$ is perpendicular to a line E and rotates about E from a given initial position while its origin 0 moves along E. The movement is such that when $[0,\infty)$ has rotated through an angle θ, the origin is at a distance $d = \sin^2(\theta/2)$ from its initial position on E. Verify that by removing the line E from the image of the rotating line, we obtain a regular surface. If the movement were such that $d = \sin(\theta/2)$, what else would need to be excluded to have a regular surface?

Solution 14.

Exercise 15 (*). Let two points $p(t)$ and $q(t)$ move with the same speed, p starting from $(0,0,0)$ and moving along the z-axis, and q starting at $(a,0,0)$, $a \neq 0$, and moving parallel to the y-axis. Show that the line through $p(t)$ and $q(t)$ describes a set in $\mathbb{R}^3$ given by

$$y(x-a) + zx = 0.$$

Is this a regular surface?

Solution 15.

Exercise 16. One way to define a system of coordinates for the sphere S^2, given by

$$x^2 + y^2 + (z-1)^2 = 1,$$

is to consider the so-called stereographic projection

$$\pi : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$$

which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy-plane with the straight line which connects N to p (Fig. 2–12). Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 \setminus \{N\}$ and (u, v) lies in the xy-plane.

a. Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$x = \frac{4u}{u^2+v^2+4}, \qquad y = \frac{4v}{u^2+v^2+4}, \qquad z = \frac{2(u^2+v^2)}{u^2+v^2+4}.$$

b. Show that it is possible, using stereographic projection, to cover the sphere with two coordinate neighborhoods.

Solution 16.

a. Let's construct the map $\pi : S^2 \to \mathbb{R}^2$ explicitly. For a point $p = (x, y, z) \in S^2 \setminus \{N\}$, the line connecting N and p can be parametrized as

$$L(t) = N + t(p - N) = (0,0,2) + t(x, y, z-2) = (tx, ty, 2 + t(z-2)) \qquad (1)$$

The intersection of this line with the xy-plane occurs when $z = 0$, so $t = 2/(2-z)$. Substituting this back to equation (1) gives

$$\pi(p) = (u, v) = \left(\frac{2x}{2-z}, \frac{2y}{2-z}\right).$$

Solving for (x, y) gives

$$(x, y) = \left(\frac{u(2-z)}{2}, \frac{v(2-z)}{2}\right).$$

From the equation for the sphere, we have

$$\left(\frac{u(2-z)}{2}\right)^2 + \left(\frac{v(2-z)}{2}\right)^2 + (z-1)^2 = 1 \implies z = \frac{2(u^2+v^2)}{u^2+v^2+4},$$

hence

$$x = \frac{4u}{u^2+v^2+4}, \quad y = \frac{4v}{u^2+v^2+4}, \quad z = \frac{2(u^2+v^2)}{u^2+v^2+4}.$$

b. Using the inverse stereographic projection π^{-1}, we can cover the whole sphere except the north pole N. To cover the north pole, use another stereographic projection from the south pole $S = (0, 0, 0)$ to the xy-plane, with the inverse map given by

$$x = \frac{4u}{u^2+v^2+4}, \quad y = \frac{4v}{u^2+v^2+4}, \quad z = \frac{8}{u^2+v^2+4}.$$

Exercise 17. Define a regular curve in analogy with a regular surface. Prove that

a. The inverse image of a regular value of a differentiable function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a regular plane curve. Give an example of such a curve which is not connected.

b. The inverse image of a regular value of a differentiable map $F : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ is a regular curve in $\mathbb{R}^3$. Show the relationship between this proposition and the classical way of defining a curve in $\mathbb{R}^3$ as the intersection of two surfaces.

***c.** The set $C = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 = y^3\}$ is not a regular curve.

Solution 17.

Exercise 18 (*). Suppose that

$$f(x, y, z) = u = \text{const.}, \qquad g(x, y, z) = v = \text{const.}, \qquad h(x, y, z) = w = \text{const.},$$

describe three families of regular surfaces and assume that at (x_0, y_0, z_0) the Jacobian

$$\frac{\partial(f, g, h)}{\partial(x, y, z)} \neq 0.$$

Prove that in a neighborhood of (x_0, y_0, z_0) the three families will be described by a mapping $F(u, v, w) = (x, y, z)$ of an open set of $\mathbb{R}^3$ into $\mathbb{R}^3$, where a local parametrization for the surface of the family $f(x, y, z) = u$, for example, is obtained by setting $u = \text{const.}$ in this mapping.

Determine F for the case where the three families of surfaces are:

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 = u = \text{const.} && \text{(spheres with center } (0, 0, 0));\\ g(x, y, z) &= \frac{y}{x} = v = \text{const.} && \text{(planes through the } z\text{-axis)};\\ h(x, y, z) &= \frac{x^2 + y^2}{z^2} = w = \text{const.} && \text{(cones with vertex at } (0, 0, 0)). \end{aligned}$$

Solution 18.

Exercise 19 (*).

Let $\alpha : (-3, 0) \to \mathbb{R}^2$ be defined by (Fig. 2–13)

$$\alpha(t) = \begin{cases} (0, \ -(t + 2)), & t \in (-3, -1), \\ \text{a regular parametrized curve joining } p = (0, -1) \text{ to } q = \left(\dfrac{1}{\pi}, 0\right), & t \in (-1, -\frac{1}{\pi}), \\ (-t, \ \sin \frac{1}{t}), & t \in \left(-\frac{1}{\pi}, \ 0\right). \end{cases}$$

It is possible to define the curve joining p to q so that all the derivatives of α are continuous at the corresponding points and α has no self-intersections. Let C be the trace of α.

a. Is C a regular curve?

b. Let a normal line to the plane $\mathbb{R}^2$ run through C so that it describes a “cylinder” S. Is S a regular surface?

Solution 19.

a. Let C be the trace of α, α is said to be regular if at every point $p \in C$, C is the graph of a C^1 function $y = f(x)$ or $x = g(y)$ in a neighborhood of p. Notice that the origin $(0, 0)$ belongs to the trace of α since $\alpha(-2) = (0, 0)$. Consider the sequence $t_n = -\frac{1}{2n\pi}$, which satisfies $t_n \in (-\frac{1}{\pi}, 0)$ for all $n \in \mathbb{N}$. Therefore, in any neighborhood of $(0, 0)$, we can find some $n \in \mathbb{N}$ such that $\alpha(t_n) \in U$, so C cannot be the graph of $x = f(y)$ locally. Similarly, C cannot be the graph of $y = g(x)$ on the line segment $\{0\} \times (-1, 1) \subseteq \mathbb{R}^2$. Hence, C is not a regular curve.

b. If the surface S were regular, then by Do Carmo Proposition 2.2.3, there exists a neighborhood V of any $p \in S$ such that V is the graph of a differentiable function $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$. However, consider a point $p \in (-\frac{1}{\pi}, 0, z)$ on the side boundary of S. In (a) we concluded that locally around $(0, 0, z)$, the curve (translated by some z along the z axis) is not the graph of a C^1 function $x = g(y, z)$ or $y = h(x, z)$, while z cannot be a function of x, y. Therefore, S is not a regular surface.

1.3 Chapter 2.3

Definition 3 (differentiability on a surface)**.** Let $f : V \subseteq S \to \mathbb{R}$ be a function defined on an open set V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subseteq V$, the composition $f \circ \mathbf{x}$ is differentiable at $\mathbf{x}^{-1}(p) \in U$. f is differentiable in V if it is differentiable at every point $p \in V$.

Definition 4 (diffeomorphism)**.** Two regular surfaces S_1 and S_2 are said to be diffeomorphic if there exists a differentiable map $\phi : S_1 \to S_2$ with a differentiable inverse $\phi^{-1} : S_2 \to S_1$. Such a map ϕ is called a diffeomorphism from S_2 to S_1.

Remark. The natural notion of equivalence associated with differentiability is the notion of diffeomorphism. From the point of view of differentiability, two diffeomorphic surfaces are indistinguishable.

Proposition 5 (change of parameters)**.** Let $p \in S$ be a point of a regular surface, and let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$, $\mathbf{y} : V \subseteq \mathbb{R}^2 \to S$ be two parametrizations with $p \in \mathbf{x}(U) \cap \mathbf{y}(V) \equiv W$. Then the change of coordinates map $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism.

Exercise 20 (*). Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A : S^2 \to S^2$ be the (antipodal) map

$$A(x, y, z) = (-x, -y, -z).$$

Prove that A is a diffeomorphism.

Solution 20. To show that A is differentiable, we need to show that for an atlas (a collection of charts that cover the surface) $(\phi_\alpha : U_\alpha \to V_\alpha)_{\alpha \in J}$ of S^2, the maps

$$\phi_\beta^{-1} \circ A \circ \phi_\alpha : \phi_\alpha^{-1}(U_\alpha \cap U_\beta \cap A^{-1}(U_\alpha \cap U_\beta)) \to \phi_\beta^{-1}(U_\alpha \cap U_\beta \cap A(U_\alpha \cap U_\beta))$$

are differentiable. Recall that S^2 has an atlas consisting of the stereographic projections from the north and south poles:

$$\{(\phi_S : U_S \to V_S)\,,\ (\phi_N : U_N \to V_N)\}\,.$$

Let the sphere be centered about $(0, 0, 0)$ with the north and south poles at $(0, 0, 1)$ and $(0, 0, -1)$, respectively. Then

$$\phi_S(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \quad \phi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

It is trivial to check that the maps

$$\phi_S^{-1} \circ A \circ \phi_N(x, y) = (-x, -y) = \phi_N^{-1} \circ A \circ \phi_S(x, y),$$
$$\phi_N^{-1} \circ A \circ \phi_N = \left(-\frac{x^2}{x^2+y^2}, -\frac{y^2}{x^2+y^2}\right) = \phi_S^{-1} \circ A \circ \phi_S(x, y)$$

are differentiable for all $(x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Since $A^{-1} = A$, the same analysis applies to A^{-1}, and thus A is a diffeomorphism.

Exercise 21. Let $S \subset \mathbb{R}^3$ be a regular surface and let $\pi : S \to \mathbb{R}^2$ be the map which takes each $p \in S$ to its orthogonal projection onto

$$\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ z = 0\}.$$

Is π differentiable?

Solution 21. The map π is differentiable if for each $p \in S$, there exists a parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U)$ such that the composition $\pi \circ \mathbf{x} : U \to \mathbb{R}^2$ is differentiable.

Given the standard basis $\{e_j\}$ of $\mathbb{R}^3$, we may assume that $N(p) = e_3$, where $N(p)$ is the normal vector of S at p. There exists a neighborhood V of p in S such that V is the graph of a differentiable function of the form $z = f(x, y)$. Thus, we can choose a parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ defined by $\mathbf{x}(u, v) = ue_1 + ve_2 + f(u, v)e_3$. By applying translations in $\mathbb{R}^3$ and U, we can ensure that $p = 0$ and $\mathbf{x}(0, 0) = 0$, and $T_p(S) = \mathbb{R}e_1 + \mathbb{R}e_2$. Then

$$\pi \circ \mathbf{x} : U \to \mathbb{R}^2, \quad (u, v) \mapsto ue_1 + ve_2$$

is differentiable for all $p \in \mathbb{R}^3$. Also, note that $\mathrm{d}\pi$ is injective, since in this coordinate we have

$$\mathrm{d}\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 22. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

Solution 22. Consider the map $\phi : \mathbb{R}^2 \to S$ defined by $\phi(u, v) = (u, v, u^2 + v^2)$. It is easy to see that ϕ is differentiable and one-to-one. The inverse map $\phi^{-1} : S \to \mathbb{R}^2$ is given by $\phi^{-1}(x, y, z) = (x, y)$, which is also differentiable. Thus, the paraboloid is diffeomorphic to the plane.

Exercise 23. Construct a diffeomorphism between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the sphere

$$x^2 + y^2 + z^2 = 1.$$

Solution 23. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\phi(x, y, z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right).$$

It is easy to see that ϕ is differentiable and one-to-one. The inverse map $\phi^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\phi^{-1}(x, y, z) = (ax, by, cz),$$

which is also differentiable. Thus, the ellipsoid is diffeomorphic to the sphere.

Exercise 24 (*). *Let $S \subset \mathbb{R}^3$ be a regular surface, and define $d : S \to \mathbb{R}$ by

$$d(p) = |p - p_0|,$$

where $p \in S$, $p_0 \in \mathbb{R}^3$, and $p_0 \notin S$. That is, d is the distance from p to a fixed point p_0 not in S. Prove that d is differentiable.

Solution 24. By definition 3, it suffices to show that for any parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, the composition $d \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable. Since S is a regular surface, for any point $p \in S$, there exists a neighborhood $V \subseteq \mathbb{R}^3$ of p such that $V \cap S$ is the graph of a differentiable function $z(x, y)$ or $x(y, z)$ or $y(x, z)$. Assume that $V \cap S$ is the graph of a differentiable function $z(x, y)$, then define a parametrization

$$\mathbf{x}(u, v) = (u, v, z(u, v)), \quad (u, v) \in U \subseteq \mathbb{R}^2,$$

where U is open in $\mathbb{R}^2$. The composition $d \circ \mathbf{x} : U \to \mathbb{R}$ is given by

$$\begin{aligned}(d \circ \mathbf{x})(u,v) = d(\mathbf{x}(u,v)) &= \sqrt{\langle \mathbf{x}(u,v) - p_0,\, \mathbf{x}(u,v) - p_0 \rangle} \\ &= \sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u,v) - z_0)^2}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial}{\partial u}(d \circ \mathbf{x})(u,v)\bigg|_{(u,v)} &= \frac{(u - x_0 + (z(u,v) - z_0)z_u(u,v))}{\sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u,v) - z_0)^2}}, \\ \frac{\partial}{\partial v}(d \circ \mathbf{x})(u,v)\bigg|_{(u,v)} &= \frac{(v - y_0 + (z(u,v) - z_0)z_v(u,v))}{\sqrt{(u - x_0)^2 + (v - y_0)^2 + (z(u,v) - z_0)^2}},\end{aligned}$$

and $z(u,v)$ is differentiable, we conclude that $d \circ \mathbf{x}$ is differentiable except when $(u,v) = (x_0, y_0) = \mathbf{x}^{-1}(p_0)$. Since the choice of $p \in S$ is arbitrary, we conclude that d is differentiable on $S \backslash \{p_0\}$.

Exercise 25. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Solution 25.

Definition 5 (differentiable map between surfaces)**.** A map $\phi : S_1 \to S_2$ between two regular surfaces is said to be <u>differentiable</u> at $p \in S_1$ if for some parametrizations $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to S_1$ and $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to S_2$ with $p \in \mathbf{x}_1(U_1)$ and $\phi(p) \in \mathbf{x}_2(U_2)$, the composition map $\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1$ is differentiable at $\mathbf{x}_1^{-1}(p)$.

Suppose that $\phi : S_1 \to S_2$ is differentiable at $p \in S_1$ with respect to parametrizations $\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to S_1$ and $\mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to S_2$. Then $\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1$ is differentiable at $q = \mathbf{x}_1^{-1}(p)$. Let $\mathbf{y}_1 : V_1 \subseteq \mathbb{R}^2 \to S_1$ and $\mathbf{y}_2 : V_2 \subseteq \mathbb{R}^2 \to S_2$ be another pair of parametrizations with $p \in \mathbf{y}_1(V_1)$ and $\phi(p) \in \mathbf{y}_2(V_2)$. Then the map

$$\mathbf{y}_2^{-1} \circ \phi \circ \mathbf{y}_1 = (\mathbf{y}_2^{-1} \circ \mathbf{x}_2) \circ (\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1) \circ (\mathbf{x}_1^{-1} \circ \mathbf{y}_1)$$

is differentiable at q since $\mathbf{y}_2^{-1} \circ \mathbf{x}_2$ and $\mathbf{x}_1^{-1} \circ \mathbf{y}_1$ are change of coordinates maps, which are diffeomorphisms by proposition 5. Conversely, suppose $\mathbf{y}_2^{-1} \circ \phi \circ \mathbf{y}_1$ is differentiable at q, then by the same argument $\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1$ is also differentiable at q. Thus, the definition of differentiability of a map between surfaces does not depend on the choice of parametrizations.

Exercise 26. Prove that the relation "S_1 is diffeomorphic to S_2" is an equivalence relation in the set of regular surfaces.

Solution 26. We verify the three properties of an equivalence relation. When S_1 is diffeomorphic S_2, we write $S_1 \cong S_2$.

(i) **Reflexivity:** For any regular surface S, the identity map $\mathrm{id}_S : S \to S$ is a diffeomorphism since it is differentiable and its inverse (itself) is also differentiable. Thus, $S \cong S$.

(ii) **Symmetry:** Suppose S_1 and S_2 are regular surfaces such that $S_1 \cong S_2$. Then there exists a diffeomorphism $\phi : S_1 \to S_2$ with differentiable inverse $\phi^{-1} : S_2 \to S_1$. Therefore, $S_2 \cong S_1$.

(iii) **Transitivity:** Suppose S_1, S_2, and S_3 are regular surfaces such that $S_1 \cong S_2$ and $S_2 \cong S_3$. Then $S_1 \cong S_3$ since composition of differentiable maps are differentiable, and composition of bijections are bijective.

Exercise 27 (*). Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and $H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$. Denote by $N = (0,0,1)$ and $S = (0,0,-1)$ the north and south poles of S^2, respectively, and let $F : S^2 - \{N\} \cup \{S\} \to H$ be defined as follows:

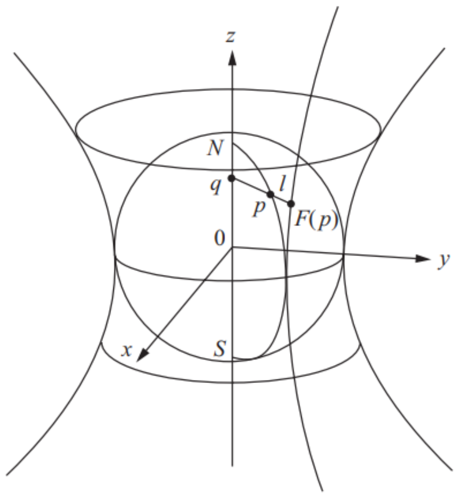

Figure 1:

For each $p \in S^2 - \{N\} \cup \{S\}$, let the perpendicular from p to the z-axis meet Oz at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (see Fig. 2–20).

Prove that F is differentiable.

Solution 27. The map F is a projection of the sphere onto a one-sheeted hyperboloid along lines parallel to the Oxy plane. Consider (1) the parametrization of the sphere

$$\mathbf{x}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

for $\theta \in [0, 2\pi)$, $\phi \in (0, \pi)$, and (2) the parametrization for the hyperboloid

$$\mathbf{y}(u, v) = (\sqrt{1 + v^2} \cos u, \sqrt{1 + v^2} \sin u, v)$$

Then we have

$$(\mathbf{y}^{-1} \circ F \circ \mathbf{x})(\theta, \phi) = \mathbf{y}^{-1} \left(\sqrt{1 + \cos^2 \phi} \cos\theta,\ \sqrt{1 + \cos^2 \phi} \sin\theta, \cos\phi \right) = (u, v).$$

Since $\mathbf{y}^{-1} \circ F \circ \mathbf{x}$ is differentiable for the parametrization $\mathbf{x}$, $\mathbf{y}$, F is differentiable.

Exercise 28.

a. Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.

b. Show that the map $E : \mathbb{R} \to S^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$ given by

$$E(t) = (\cos t, \sin t), \quad t \in \mathbb{R},$$

is differentiable (geometrically, E "wraps" $\mathbb{R}$ around S^1).

Solution 28.

a. Suppose $\alpha : I \to \mathbb{R}^2$ is a regular curve with trace $C \subseteq \mathbb{R}^2$. We say that a function $f : C \to \mathbb{R}$ is differentiable along C if the composition $f \circ \alpha : I \to \mathbb{R}$ is differentiable. In other words, we need to check that the derivative $(f \circ \alpha)'(t)$ exists for all $t \in I$.

b. *

Exercise 29. Let C be a plane regular curve which lies on one side of a straight line r of the plane and meets r at the points p, q (see Fig. 2–21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

Solution 29. We can analyze the point $p \in C$ locally. Assume that r is the z axis, and C is the graph of a differentiable function $y = f(x)$ in a neighborhood of p, since C is a regular curve. Since S is the surface of revolution generated by rotating C about r, we claim that there is a local chart at $p \in S$ given by

$$\mathbf{x} : U \subseteq \mathbb{R}^2 \to S, \quad (x, y) \mapsto (x, y, f(\sqrt{x^2 + y^2})),$$

where U is an open set in $\mathbb{R}^2$. We will check each condition given in definition (1) for S.

(i) $\mathbf{x}$ is differentiable. We can calculate its differential at some $(x, y) \in U$ as

$$\mathrm{d}\mathbf{x}_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{x}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) & \frac{y}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) \end{pmatrix}. \tag{2}$$

Since f is differentiable, the partial derivatives of $\mathbf{x}$ exist whenever $(x, y) \neq (0, 0)$. By symmetry, $f(w) = f(-w)$, so $f'(w) = -f'(-w)$. When $(x, y) = (0, 0)$, we have $f'(0) = 0$, and

$$\frac{x}{\sqrt{x^2+y^2}}, \quad \frac{y}{\sqrt{x^2+y^2}}$$

are bounded, so $\mathrm{d}\mathbf{x}_{(x,y)}$ exists at $(0, 0)$. To satisfy the symmetry condition, we require that f' is odd, hence f is even, and all the odd-order derivatives of f vanish at 0. Similarly, the odd-order derivatives of g such that $y = g(x)$ in a neighborhood of q must also vanish.

(ii) $\mathbf{x}$ is a homeomorphism, since the graph of a continuous function is homeomorphic to its domain.

(iii) From equation (2), we have $|\partial(x, y)/\partial(u, v)| = 1$, so $\mathrm{d}\mathbf{x}$ is one-to-one. Hence $\mathrm{d}\mathbf{x}_{(x,y)}$ is one-to-one for all $(x, y) \in U$.

Exercise 30. Prove that the rotations of a surface of revolution S about its axis are diffeomorphisms of S.

Solution 30. Let S be a surface of revolution generated by rotating a regular curve C around an axis $r \subseteq \mathbb{R}^3$, without loss of generality let r be the z-axis and let C lie on the Oxz plane. Since C is a regular curve, it can be parametrized as $(f(t), 0, g(t))$ for $t \in (a, b)$. Then S has a parametrization $\mathbf{x}(u, v) = (f(v) \cos u, f(v) \sin u, v)$ for $u \in [0, 2\pi)$ and $v \in (0, \infty)$. A rotation of S about its axis by an angle θ is given by the map

$$R_\theta : S \to S, \quad (x, y, z) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z).$$

Consider the composition $(\mathbf{y}^{-1} \circ R_\theta \circ \mathbf{x}) : U \to U$ for parametrizatoins $\mathbf{x}$, $\mathbf{y}$ of S. Then

$$\left(\mathbf{y}^{-1} \circ R_\theta \circ \mathbf{x}\right)(u, v) = \mathbf{y}^{-1}\left((f(v) \cos(u + \theta), f(v) \sin(u + \theta), v)\right) = (u + \theta, v).$$

is differentiable since $\mathbf{x}$, $\mathbf{y}$, R_θ is differentiable. Similarly, the inverse map $R_\theta^{-1} = R_{-\theta}$ is also differentiable. Thus, R_θ is a diffeomorphism of S.

Exercise 31. Parametrized surfaces are often useful to describe sets Σ which are regular surfaces except for a finite number of points and a finite number of lines. For instance, let C be the trace of a regular parametrized curve$\alpha : (a, b) \to \mathbb{R}^3$ which does not pass through the origin $O = (0, 0, 0)$. Let Σ be the set generated by the displacement of a straight line l passing through a moving point $p \in C$ and the fixed point 0 (a cone with vertex 0; see Fig. 2–22).

a. Find a parametrized surface $\mathbf{x}$ whose trace is Σ.

b. Find the points where $\mathbf{x}$ is not regular.

c. What should be removed from Σ so that the remaining set is a regular surface?

Solution 31.

Exercise 32 (*). Show that the definition of differentiability of a function $f : V \subset S \to \mathbb{R}$ given in the text (Def. 1) is equivalent to the following: <u>f is differentiable in $p \in V$ if it is the restriction to V of a differentiable function defined in an open set of $\mathbb{R}^3$ containing p.</u>

Solution 32.

Exercise 33. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is,

$$A = U \cap S,$$

where U is an open set in $\mathbb{R}^3$.

Solution 33. Suppose $A \subset S$ is a regular surface. Then there are parametrizations $\mathbf{x}_A : U_A \to A$ and $\mathbf{x}_S : U_S \to S$ from open sets U_A and U_S in $\mathbb{R}^2$. Then the map $\mathbf{x}_S^{-1} \circ \mathbf{x}_A : U_A \to U_S$ is an open map: for V open in U_A, $(\mathbf{x}_S^{-1} \circ \mathbf{x}_A)(V)$ is open in $\mathrm{dom}(\mathbf{x}_S)$ by the Inverse Function Theorem and $\mathbf{x}_S$ is a homeomorphism. Therefore, $\mathbf{x}_A(V) = \mathbf{x}_S \circ (\mathbf{x}_S^{-1} \circ \mathbf{x}_A)(V)$ is open in S. Now let $V = U_A$ be the whole domain, then $\mathbf{x}_A(U_A) = A$ is open in S.

Conversely, suppose A is open in S. Let $p \in A$, *

Exercise 34. Let C be a regular curve and let $\alpha : I \subset \mathbb{R} \to C$, $\beta : J \subset \mathbb{R} \to C$ be two parametrizations of C in a neighborhood of $p \in \alpha(I) \cap \beta(J) = W$. Let

$$h = \alpha^{-1} \circ \beta : \beta^{-1}(W) \to \alpha^{-1}(W)$$

be the change of parameters. Prove that

a. h is a diffeomorphism.

b. The absolute value of the arc length of C in W does not depend on which parametrization is chosen to define it, that is,

$$\left| \int_{t_0}^{t} |\alpha'(t)|\, dt \right| = \left| \int_{\tau_0}^{\tau} |\beta'(\tau)|\, d\tau \right|, \qquad t = h(\tau),\ t \in I,\ \tau \in J.$$

Solution 34.

a. By the chain rule and inverse function rule,

$$h'(t) = \left. \frac{1}{\alpha'(\alpha^{-1})} \right|_{\beta} \circ \beta'(t).$$

Since α and β are parametrizations for a regular curve, $|\alpha|, |\beta| \neq 0$. Then h' always exists and h is differentiable on $\beta^{-1}(W)$. Similarly, we have $h^{-1} = \beta^{-1} \circ \alpha$, so by a similar calculation we know h^{-1} is differentiable on $\alpha^{-1}(W)$. Therefore, h is a diffeomorphism.

b. For $t \in I$, $\tau \in J$, we have

$$\begin{aligned}\left|\int_{t_0}^{t} \mathrm{d}t\, |\alpha'(t)|\right| &= \left|\int_{\tau_0}^{\tau} \mathrm{d}\tau\, h'(\tau)\, |(\alpha'|_h \circ h(\tau))|\right| \\ &= \left|\int_{\tau_0}^{\tau} \mathrm{d}\tau\, \frac{1}{\alpha'(\alpha^{-1})}\Big|_\beta \circ \beta'(\tau)\, \left|\alpha' \circ (\alpha^{-1} \circ \beta)(\tau)\right|\right| \\ &= \left|\int_{\tau_0}^{\tau} |\beta'(\tau)|\, d\tau\right|.\end{aligned}$$

Exercise 35 (*). Let $R^2 = \{(x, y, z) \in \mathbb{R}^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P : \mathbb{C} \to \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \quad a_0 \neq 0,\, a_i \in \mathbb{C},\, i = 0, \ldots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F : S^2 \to S^2$ given by

$$F(p) = \begin{cases} \pi_N^{-1} \circ P \circ \pi_N(p), & \text{if } p \in S^2 - \{N\}, \\ N, & \text{if } p = N, \end{cases}$$

is differentiable.

Solution 35. Given a point $p \in S^2\backslash\{N\}$, write it as $p = (x, y, z)$. Since the composition of differentiable functions is differentiable, we only need to show that π_N, π_N^{-1} and P are differentiable. The stereographic projection $\pi_N : S^2\backslash\{N\} \to \mathbb{R}^2$ is given by

$$\pi_N(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right).$$

Since $z \neq 1$ for all $p \in S^2\backslash\{N\}$, π_N is differentiable. Similarly, note that the inverse stereographic projection $\pi_N^{-1} : \mathbb{R}^2 \to S^2\backslash\{N\}$ is given by

$$\pi_N^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

Since $u^2 + v^2 + 1 > 0$ for all $(u, v) \in \mathbb{R}^2$, π_N^{-1} is differentiable. Moreover, polynomials are differentiable everywhere, so P is differentiable. Thus, F is differentiable on $S^2\backslash\{N\}$.

1.4 Chapter 2.4

Exercise 36. Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Solution 36. Suppose 0 is a regular value of f, then by definition we have a regular surface defined implicitly by $f(x, y, z) = 0$. By proposition, a regular surface can be locally represented as the graph of a differentiable function. Without loss of generality, write $z = g(x, y) = f^{-1}(0)$ in a neighborhood of $p = (x_0, y_0, z_0)$. By the Inverse Function Theorem, we have

$$\frac{\partial g}{\partial x}(x_0, y_0) = -\frac{f_x(x_0, y_0, z_0)}{f_z(x_0, y_0, z_0)}, \quad \frac{\partial g}{\partial y}(x_0, y_0) = -\frac{f_y(x_0, y_0, z_0)}{f_z(x_0, y_0, z_0)}. \tag{3}$$

Then the tangent at p is given by

$$\begin{aligned} z &= g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) \\ &= z_0 - \left(\frac{\partial f/\partial x}{\partial f/\partial z}\right)_p (x - x_0) - \left(\frac{\partial f/\partial y}{\partial f/\partial z}\right)_p (y - y_0), \end{aligned} \tag{4}$$

and

$$(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0. \tag{5}$$

Remark. We can express this more compactly with $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{x} = (x, y, z)$ and $\nabla f = (f_x, f_y, f_z)$, then

$$(\mathbf{x} - \mathbf{x}_0) \cdot \nabla f(\mathbf{x}_0) = 0. \tag{6}$$

Exercise 37. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z-axis.

Solution 37. Use the result of Exercise 2.4.1, we have

$$2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0. \tag{7}$$

At the point $(x_0, y_0, 0)$, we have

$$2x_0(x - x_0) + 2y_0(y - y_0) = 0. \tag{8}$$

The normal vector of the tangent plane is $(2x_0, 2y_0, 0)$, which is perpendicular to the z-axis. Thus the tangent planes are all parallel to the z-axis.

Exercise 38. Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and show that the tangent plane is the graph of the differential df_p.

Solution 38. Since $z_0 = f(x_0, y_0)$, the tangent plane at $p_0 = (x_0, y_0, z_0)$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{9}$$

Recall the definition of the differential of a function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$df_{(x_0,y_0)}(h,k) = f_x(x_0,y_0)h + f_y(x_0,y_0)k. \tag{10}$$

Let $h = x - x_0$, $k = y - y_0$, then

$$df_{(x_0,y_0)}(x - x_0, y - y_0) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0). \tag{11}$$

Thus the tangent plane can be expressed as

$$z = f(x_0,y_0) + df_{(x_0,y_0)}(x - x_0, y - y_0), \tag{12}$$

which is the graph of the differential $df_{(x_0,y_0)}$.

Exercise 39. Show that the tangent planes of a surface given by $z = xf(y/x)$, $x \neq 0$, where f is a differentiable function, all pass through the origin $(0,0,0)$.

Solution 39. Let $g(x,y) = xf(y/x)$, then $z = g(x,y)$ and

$$g_x(x,y) = f(y/x) - \frac{y}{x}f'(y/x), \quad g_y(x,y) = f'(y/x).$$

Since $z_0 = x_0 f(y_0/x_0)$, the tangent plane at (x_0,y_0,z_0) is given by

$$\begin{aligned} z &= x_0 f(y_0/x_0) + g_x(x_0,y_0)(x - x_0) + g_y(x_0,y_0)(y - y_0) \\ &= x_0 f(y_0/x_0) + \left(f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0) \right)(x - x_0) + f'(y_0/x_0)(y - y_0). \end{aligned}$$

We can check that $(0,0,0)$ is a solution, hence the desired result.

Exercise 40. If a coordinate neighborhood of a regular surface can be parametrized in the form

$$\mathbf{x}(u,v) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

Solution 40. Here $\alpha_1(u), \alpha_2(v) \in \mathbb{R}^3$. The tangent plane at (u_0,v_0) is spanned by the vectors

$$\mathbf{x}_u(u_0,v_0) = \alpha_1'(u_0), \quad \mathbf{x}_v(u_0,v_0) = \alpha_2'(v_0).$$

Then the normal vector of the plane is given by

$$N(u_0,v_0) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u_0,v_0) = \frac{\alpha_1' \wedge \alpha_2'}{|\alpha_1' \wedge \alpha_2'|}(u_0,v_0).$$

*

Exercise 41. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with everywhere nonzero curvature. Consider the tangent surface of α (Example 5 of Sec. 2–3)

$$\mathbf{x}(t,v) = \alpha(t) + v\,\alpha'(t), \qquad t \in I,\ v \neq 0.$$

Show that the tangent planes along the curve $\mathbf{x}(\text{const.}, v)$ are all equal.

Solution 41.

Exercise 42. Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point of $\mathbb{R}^3$ (see Example 1 of Sec. 2–3). Show that

$$(df)_p(w) = 2\, w \cdot (p - p_0), \qquad w \in T_p(S).$$

Solution 42.

Exercise 43. Prove that if $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and $S \subset \mathbb{R}^3$ is a regular surface invariant under L, i.e., $L(S) \subset S$, then the restriction $L|_S$ is a differentiable map and

$$dL_p(w) = L(w), \qquad p \in S,\ w \in T_p(S).$$

Solution 43.

Exercise 44. Show that the parametrized surface

$$\mathbf{x}(u, v) = (v \cos u,\ v \sin u,\ au), \qquad a \neq 0,$$

is regular. Compute its normal vector $N(u, v)$ and show that along the coordinate line $u = u_0$ the tangent plane of $\mathbf{x}$ rotates about this line in such a way that the tangent of its angle with the z axis is proportional to the inverse of the distance $v(= \sqrt{x^2 + y^2})$ of the point $\mathbf{x}(u_0, v)$ to the z axis.

Solution 44.

Exercise 45 (Tubular Surfaces)**.** Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r\big(n(s)\cos v + b(s)\sin v\big), \qquad r = \text{const.} \neq 0,\ s \in I,$$

be a parametrized surface (the tube of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

$$N(s, v) = -\big(n(s)\cos v + b(s)\sin v\big).$$

Solution 45. Let $\mathbf{x} : U \to \mathbb{R}^3$ as defined in the problem statement be a regular Parametrization, where U is an open set in $\mathbb{R}^2$. The unit normal vector at each point $q \in \mathbf{x}(U)$ is defined as

$$N(q) = \frac{\mathbf{x}_s \wedge \mathbf{x}_v}{|\mathbf{x}_s \wedge \mathbf{x}_v|}(q).$$

Let prime denote derivative with respect to s. Then we have

$$\mathbf{x}_s = \alpha'(s) + r\big(n'(s)\cos v + b'(s)\sin v\big), \quad \mathbf{x}_v = r\big(-n(s)\sin v + b(s)\cos v\big),$$

and by the Frenet-Serret formulas,

$$\alpha'(s) = t(s), \quad n'(s) = -\kappa(s)t(s) - \tau(s)b(s), \quad b'(s) = \tau(s)n(s),$$

where t is the unit tangent, κ is the curvature, and τ is the torsion of α. Thus,

$$\begin{aligned}
\mathbf{x}_s &= t(s) + r\big((-\kappa(s)t(s) - \tau(s)b(s))\cos v + \tau(s)n(s)\sin v\big),\\
\mathbf{x}_v &= r\big(-n(s)\sin v + b(s)\cos v\big).
\end{aligned}$$

Now suppress s and compute the wedge product in the Frenet frame $\{t, n, b\}$:

$$\begin{aligned}
\mathbf{x}_s \wedge \mathbf{x}_v &= \big(t + r\big(-\kappa t \cos v - \tau b \cos v + \tau n \sin v\big)\big) \wedge r\big(-n \sin v + b \cos v\big) \\
&= -r(t \wedge n)\sin v + r(t \wedge b)\cos v - r^2\kappa \sin v \cos v (t \wedge n) - r^2 \kappa \cos^2 v (t \wedge b) \\
&\quad + r^2 \tau \sin v \cos v (b \wedge n) + r^2 \tau \sin v \cos v (n \wedge b) \\
&= -r(1 - r\kappa \cos v)\,(\cos v n + \sin v b)\,.
\end{aligned}$$

Dividing by the norm and noting that n and b are unit length and orthogonal, we have

$$N(s, v) = -\big(n(s)\cos v + b(s)\sin v\big).$$

Exercise 46. Show that the normals to a parametrized surface given by

$$\mathbf{x}(u, v) = \big(f(u)\cos v,\ f(u)\sin v,\ g(u)\big), \qquad f(u) \neq 0,\ g'(u) \neq 0,$$

all pass through the z axis.

Solution 46.

Exercise 47. Show that each of the equations $(a, b, c \neq 0)$

$$x^2 + y^2 + z^2 = ax, \quad x^2 + y^2 + z^2 = by, \quad x^2 + y^2 + z^2 = cz$$

define a regular surface and that they all intersect orthogonally.

Solution 47. Recall the following proposition:

Proposition 6. A surface defined implicitly by $f(x, y, z) = 0$ is a regular surface if 0 is a regular value of f, i.e., $\nabla f \neq 0$ on the surface.

Let $f_1(x, y, z) = x^2+y^2+z^2-ax$, $f_2(x, y, z) = x^2+y^2+z^2-by$, $f_3(x, y, z) = x^2+y^2+z^2-cz$. Then we have

$$\nabla f_1 = (2x - a, 2y, 2z), \quad \nabla f_2 = (2x, 2y - b, 2z), \quad \nabla f_3 = (2x, 2y, 2z - c). \tag{13}$$

Since $a, b, c \neq 0$, we have $\nabla f_1 = 0$ implies $(x, y, z) = (a/2, 0, 0)$, which does not satisfy the equation of the surface. Similarly, we can show that $\nabla f_2 \neq 0$ and $\nabla f_3 \neq 0$ on the surfaces. Thus all three surfaces are regular surfaces. Moreover, the normal vectors of the tangent planes at (x, y, z) are given by ∇f_1, ∇f_2, and ∇f_3, respectively. Then we have

$$\begin{aligned}
\nabla f_1 \cdot \nabla f_2 &= 4x^2 + 4y^2 + 4z^2 - 2ax - 2by = 0, \\
\nabla f_2 \cdot \nabla f_3 &= 4x^2 + 4y^2 + 4z^2 - 2by - 2cz = 0, \\
\nabla f_1 \cdot \nabla f_3 &= 4x^2 + 4y^2 + 4z^2 - 2ax - 2cz = 0.
\end{aligned} \tag{14}$$

Hence they all intersect orthogonally.

Exercise 48. A critical point of a differentiable function $f : S \to \mathbb{R}$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$.

- **a.** Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|$, $p \in S$, $p_0 \notin S$ (cf. Exercise 5, Sec. 2–3). Show that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.
- **b.** Let $h : S \to \mathbb{R}$ be given by $h(p) = p \cdot v$, where $v \in \mathbb{R}^3$ is a unit vector (cf. Example 1, Sec. 2–3). Show that $p \in S$ is a critical point of f if and only if v is a normal vector of S at p.

Solution 48.

Exercise 49. Let Q be the union of the three coordinate planes $x = 0$, $y = 0$, $z = 0$. Let $p = (x, y, z) \in \mathbb{R}^3 - Q$.

a. Show that the equation in t,

$$\frac{x^2}{a-t} + \frac{y^2}{b-t} + \frac{z^2}{c-t} \equiv f(t) = 1, \qquad a > b > c > 0,$$

has three distinct real roots t_1, t_2, t_3.

b. Show that for each $p \in \mathbb{R}^3 - Q$, the sets given by

$$f(t_1) - 1 = 0, \quad f(t_2) - 1 = 0, \quad f(t_3) - 1 = 0$$

are regular surfaces passing through p which are pairwise orthogonal.

Solution 49.

a. Consider the function F which implicitly defines the surfaces:

$$F(t) = \frac{x^2}{a-t} + \frac{y^2}{b-t} + \frac{z^2}{c-t} - 1.$$

It has three vertical asymptotes at $t = a, b, c$. Moreover, it is continuous and monotone increasing in each of the open intervals $t \in (-\infty, c)$, (c, b), (b, a), (a, ∞). Thus by the Intermediate Value Theorem, there exist exactly three distinct real roots $t_1 < c < t_2 < b < t_3 < a$.

b. Given $F(t_j)$ and $F(t_k)$, their point of intersection p is

$$\begin{aligned} 0 &= F(t_j) - F(t_k) \\ &= \left(\frac{x^2}{a-t_j} + \frac{y^2}{b-t_j} + \frac{z^2}{c-t_j} - 1\right) - \left(\frac{x^2}{a-t_k} + \frac{y^2}{b-t_k} + \frac{z^2}{c-t_k} - 1\right) \\ &= (t_j - t_k)\left(\frac{x^2}{(a-t_j)(a-t_k)} + \frac{y^2}{(b-t_j)(b-t_k)} + \frac{z^2}{(c-t_j)(c-t_k)}\right). \end{aligned}$$

Then, assuming $t_j \neq t_k$, we have

$$\begin{aligned} \nabla(F(t_j)) \cdot \nabla(F(t_k)) &= 4\left(\frac{x}{(a-t_j)}, \frac{y}{(b-t_j)}, \frac{z}{(c-t_j)}\right) \cdot \left(\frac{x}{(a-t_k)}, \frac{y}{(b-t_k)}, \frac{z}{(c-t_k)}\right) \\ &= 4\left(\frac{x^2}{(a-t_j)(a-t_k)} + \frac{y^2}{(b-t_j)(b-t_k)} + \frac{z^2}{(c-t_j)(c-t_k)}\right) = 0. \end{aligned}$$

Therefore, the surfaces intersect orthogonally for all $p \in \mathbb{R}^3$, i.e. they are pairwise orthogonal. Furthermore, we see $\nabla F|_p = 0$ if and only if $p = 0 \notin S$, so by proposition 6, they are regular surfaces.

Remark. The surfaces described above are called confocal quadrics. Any point $(x_0, y_0, z_0) \in \mathbb{R}^3 \backslash \{0\}$ lies on exactly one surface of each of the three types of confocal quadratics, and the three quadrics intersect orthogonally at that point.

Exercise 50. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Solution 50.

Exercise 51. Let w be a tangent vector to a regular surface S at a point $p \in S$ and let $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(\bar{u},\bar{v})$ be two parametrizations at p. Suppose that the expressions of w in the bases associated to $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(\bar{u},\bar{v})$ are

$$w = \alpha_1 \mathbf{x}_u + \alpha_2 \mathbf{x}_v$$

and

$$w = \beta_1 \bar{\mathbf{x}}_{\bar{u}} + \beta_2 \bar{\mathbf{x}}_{\bar{v}}.$$

Show that the coordinates of w are related by

$$\beta_1 = \alpha_1 \frac{\partial \bar{u}}{\partial u} + \alpha_2 \frac{\partial \bar{u}}{\partial v}, \qquad \beta_2 = \alpha_1 \frac{\partial \bar{v}}{\partial u} + \alpha_2 \frac{\partial \bar{v}}{\partial v},$$

where $\bar{u} = \bar{u}(u,v)$ and $\bar{v} = \bar{v}(u,v)$ are the expressions of the change of coordinates.

Solution 51.

Exercise 52 (*). Two regular surfaces S_1 and S_2 intersect transversally if whenever $p \in S_1 \cap S_2$ then $T_p(S_1) \neq T_p(S_2)$. Prove that if S_1 intersects S_2 transversally, then $S_1 \cap S_2$ is a regular curve.

Solution 52. Let S_1, S_2 be two regular surfaces that intersect transversally, and let $p \in S_1 \cap S_2$. Since S_1, S_2 are regular surfaces, there exists a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ and a neighborhood V_1 of p such that $S_1 \cap V_1 = f^{-1}(0) \cap V_1$. Similarly, there exists a differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$ and a neighborhood V_2 of p such that $S_2 \cap V_2 = g^{-1}(0) \cap V_2$. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $F(q) = (f(q), g(q))$. Then

$$F^{-1}(0,0) = f^{-1}((0,0)) \cap g^{-1}((0,0)) \supseteq (V_1 \cap V_2) \cap (S_1 \cap S_2).$$

Let $V = V_1 \cap V_2$. In V, we have $S_1 \cap S_2 = F^{-1}(0,0)$. Since $T_p(S_1) \neq T_p(S_2)$, we have $N_{p_1}(0,0) \wedge N_{p_2}(0,0) \neq 0$, where

$$N_{p_1} = \frac{(f_x, f_y, f_z)(p)}{\|(f_x, f_y, f_z)(p)\|}, \quad N_{p_2} = \frac{(g_x, g_y, g_z)(p)}{\|(g_x, g_y, g_z)(p)\|}.$$

Hence

$$\mathrm{d}F_{(x,y,z)} = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix} (x,y,z) \neq 0,$$

and $\mathrm{d}F$ is surjective. Therefore, $(0,0)$ is a regular point of F, and by [Do Carmo] Problem 2.2.17 (b) (The inverse image of a regular value of a differentiable map $F : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ is a regular curve in $\mathbb{R}^3$), $S_1 \cap S_2$ is a regular curve.

Exercise 53. Prove that if a regular surface S meets a plane P in a single point p, then this plane coincides with the tangent plane of S at p.

Solution 53.

Exercise 54. Let $S \subset \mathbb{R}^3$ be a regular surface and $P \subset \mathbb{R}^3$ be a plane. If all points of S are on the same side of P, prove that P is tangent to S at all points of $P \cap S$.

Solution 54.

Exercise 55 **(*).** Show that the perpendicular projections of the center $(0, 0, 0)$ of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

onto its tangent planes constitute a regular surface given by

$$\{(x, y, z) \in \mathbb{R}^3;\ (x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2\} - \{(0, 0, 0)\}.$$

Solution 55.

Exercise 56 **(*).** Let $f : S \to \mathbb{R}$ be a differentiable function on a connected regular surface S. Assume that $df_p = 0$ for all $p \in S$. Prove that f is constant on S.

Solution 56. Let $p, q \in S$ be two arbitrary points. Since S is connected, there exists a piecewise regular curve $\alpha : [0, 1] \to S$ such that $\alpha(0) = p$, $\alpha(1) = q$. Then we have

$$(f \circ \alpha)'(t) = df_{\alpha(t)}(\alpha'(t)) = 0, \quad t \in [0, 1].$$

Thus $f \circ \alpha$ is constant on $[0, 1]$, and in particular, we have

$$f(p) = f(\alpha(0)) = f(\alpha(1)) = f(q).$$

Since p, q are arbitrary points in S, we conclude that f is constant on S.

Exercise 57. Prove that if all normal lines to a connected regular surface S meet a fixed straight line, then S is a piece of a surface of revolution.

Solution 57.

Exercise 58. Prove that the map $F : S^2 \to S^2$ defined in Exercise 16 of Sec. 2-3 has only a finite number of critical points (see Exercise 13).

Solution 58. From Problem 2.3.16, $F : S^2 \to S^2$ is differentiable. Let $p \in S^2$ be a critical point of F, then $\mathrm{d}F_p = 0$. Since $F = \pi_N^{-1} \circ P \circ \pi_N$, by the chain rule, we have

$$\mathrm{d}F_p = \mathrm{d}(\pi_N^{-1})_{P(\pi_N(p))} \circ \mathrm{d}P_{\pi_N(p)} \circ \mathrm{d}(\pi_N)_p.$$

Note that $\mathrm{d}(\pi_N)_p$ and $\mathrm{d}(\pi_N^{-1})_{P(\pi_N(p))}$ are isomorphisms, so $\mathrm{d}F_p = 0$ if and only if $\mathrm{d}P_{\pi_N(p)} = 0$. Since $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree n, $\mathrm{d}P$ is a polynomial of degree $n - 1$, and thus has $n - 1$ roots by the Fundamental Theorem of Algebra. Therefore, the map $F : S^2 \to S^2$ has only a finite number of critical points.

Exercise 59 (Chain Rule)**.** Show that if $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are differentiable maps and $p \in S_1$, then

$$d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p.$$

Solution 59.

Exercise 60. Prove that if two regular curves C_1 and C_2 of a regular surface S are tangent at a point $p \in S$, and if $\varphi : S \to S$ is a diffeomorphism, then $\varphi(C_1)$ and $\varphi(C_2)$ are regular curves which are tangent at $\varphi(p)$.

Solution 60.

Exercise 61. Show that if p is a point of a regular surface S, it is possible, by a convenient choice of the (x, y, z) coordinates, to represent a neighborhood of p in S in the form $z = f(x, y)$ so that

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0.$$

(This is equivalent to taking the tangent plane to S at p as the xy plane.)

Solution 61.

Exercise 62 (Theory of Contact)**.** Two regular surfaces, S and $\bar{S}$, in $\mathbb{R}^3$, which have a point p in common, are said to have contact of order ≥ 1 at p if there exist parametrizations with the same domain $\mathbf{x}(u, v)$, $\bar{\mathbf{x}}(u, v)$ at p of S and $\bar{S}$, respectively, such that $\mathbf{x}_u = \bar{\mathbf{x}}_u$ and $\mathbf{x}_v = \bar{\mathbf{x}}_v$ at p. If, moreover, some of the second partial derivatives are different at p, the contact is said to be of order exactly equal to 1. Prove that

- **a.** The tangent plane $T_p(S)$ of a regular surface S at the point p has contact of order ≥ 1 with the surface at p.
- **b.** If a plane has contact of order ≥ 1 with a surface S at p, then this plane coincides with the tangent plane to S at p.
- **c.** Two regular surfaces have contact of order ≥ 1 if and only if they have a common tangent plane at p, i.e., they are tangent at p.
- **d.** If two regular surfaces S and $\bar{S}$ of $\mathbb{R}^3$ have contact of order ≥ 1 at p and if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism of $\mathbb{R}^3$, then the images $F(S)$ and $F(\bar{S})$ are regular surfaces which have contact of order ≥ 1 at $F(p)$ (that is, the notion of contact of order ≥ 1 is invariant under diffeomorphisms).
- **e.** If two surfaces have contact of order ≥ 1 at p, then $\lim_{r \to 0} \frac{d}{r} = 0$, where d is the length of the segment which is determined by the intersections with the surfaces of some parallel to the common normal, at a distance r from this normal.

Solution 62.

Exercise 63 (Do Carmo 2.4.28)**.**

- **a.** Define regular value for a differentiable function $f : S \to \mathbb{R}$ on a regular surface S.
- **b.** Show that the inverse image of a regular value of a differentiable function on a regular surface S is a regular curve on S.

Solution 63.

- **a.** A regular value of a differentiable function $f : S \to \mathbb{R}$ defined on a regular surface S is a value $c \in \mathbb{R}$ such that for every point $p \in f^{-1}(c)$, the differential $\mathrm{d}f_p : T_p(S) \to \mathbb{R}$ is surjective (i.e., $\mathrm{d}f_p \neq 0$).
- **b.** Let c be a regular value of a differentiable function $f : S \to \mathbb{R}$ and let $p \in f^{-1}(c)$. Pick a local parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ such that $\mathbf{x}((0,0)) = p$. Define $g : U \to \mathbb{R}$ by $g = f \circ \mathbf{x}$, then $g(0,0) = f(\mathbf{x}(0,0)) = f(p) = c$. Since $\mathrm{d}f_p \neq 0$ and $\mathrm{d}\mathbf{x}_{(0,0)}$ is surjective onto T_pS, we have $\mathrm{d}g_{(0,0)} \neq 0$. By the Implicit Function Theorem, there exists a neighborhood $V \subseteq U$ of

$(0,0)$ such that $g^{-1}(c) \cap V$ is the graph of a C^1 function, say $v = \phi(u)$. Then we can define a local parametrization of the curve $f^{-1}(c)$ on S by

$$\alpha(u) = \mathbf{x}(u, \phi(u)), \quad u \in I$$

where I is some neighborhood of $u = 0$. Suppose for some u^*, we have $\alpha'(u^*) = 0$, then

$$\mathrm{d}\mathbf{x}_{(u^*, \phi(u^*))}\,(1, \phi'(u^*)) = 0.$$

Since $\mathrm{d}\mathbf{x}$ is one-to-one, we must have $(1, \phi'(u^*)) = 0$, contradiction. Thus, $\alpha'(u) \neq 0$ for all $u \in I$, and in a neighborhood of each $p \in f^{-1}(c)$, $f^{-1}(c)$ is the image of a regular curve α on S. Patching the local parametrizations together, we conclude that $f^{-1}(c)$ is a regular curve on S.

1.5 Chapter 2.5

Definition 6 (first fundamental form)**.** The quadratic form $I_p : T_p(S) \to \mathbb{R}$ defined by $I_p(w) = \langle w, w\rangle_p = |w|^2$ is called the first fundamental form of the surface S at the point $p \in S$.

Definition 7 (area)**.** Let $R \subseteq S$ be a bounded region of a regular surface S contained in the coordinate neighborhood of a parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$. The positive number

$$A(R) = \iint_{\mathbf{x}^{-1}(R)} \mathrm{d}u\,\mathrm{d}v\,|\mathbf{x}_u \wedge \mathbf{x}_v| = \iint_{\mathbf{x}^{-1}(R)} \mathrm{d}u\,\mathrm{d}v\,\sqrt{EG - F^2} \tag{15}$$

is called the area of the region R.

Exercise 64. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

a. $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.

b. $\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

c. $\mathbf{x}(u, v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

d. $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

Solution 64.

Exercise 65. Let $\mathbf{x}(\varphi, \theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ be a parametrization of the unit sphere S^2. Let P be the plane $x = z \cot\alpha$, $0 < \alpha < \pi$, and β be the acute angle which the curve $P \cap S^2$ makes with the semimeridian $\varphi = \varphi_0$. Compute $\cos\beta$.

Solution 65.

Exercise 66. Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2-2).

Solution 66. Refer to Exercise 2.2.16, let the sphere be $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+(z-1)^2 = 1\}$. The stereographic projection from the north pole $N = (0, 0, 2)$ to the xy-plane is given by

$$\mathbf{x}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}\right).$$

We have

$$\begin{aligned}
\mathbf{x}_u &= \left(\frac{4(-u^2 + v^2 + 4)}{(u^2 + v^2 + 4)^2}, \frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{16u}{(u^2 + v^2 + 4)^2}\right),\\
\mathbf{x}_v &= \left(\frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{4(u^2 - v^2 + 4)}{(u^2 + v^2 + 4)^2}, \frac{16v}{(u^2 + v^2 + 4)^2}\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle &= \frac{16(-u^2 + v^2 + 4)^2 + 64u^2v^2 + 256u^2}{(u^2 + v^2 + 4)^4} = \frac{16}{(u^2 + v^2 + 4)^2},\\
F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle &= \frac{-32uv(-u^2 + v^2 + 4) - 32uv(u^2 - v^2 + 4) + 256uv}{(u^2 + v^2 + 4)^4} = 0,\\
G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle &= \frac{64u^2v^2 + 16(u^2 - v^2 + 4)^2 + 256v^2}{(u^2 + v^2 + 4)^4} = \frac{16}{(u^2 + v^2 + 4)^2}.
\end{aligned}$$

Therefore, the first fundamental form is

$$I_p = \frac{16}{(u^2+v^2+4)^2}\left((u')^2+(v')^2\right).$$

Exercise 67. Given the parametrized surface

$$\mathbf{x}(u,v) = (u\cos v, u\sin v, \log\cos v + u), \quad -\frac{\pi}{2} < v < \frac{\pi}{2},$$

show that the two curves $\mathbf{x}(u,v_1)$, $\mathbf{x}(u,v_2)$ determine segments of equal lengths on all curves $\mathbf{x}(u,\text{const.})$.

Solution 67.

Exercise 68. Show that the area A of a bounded region R of the surface $z = f(x,y)$ is

$$A = \iint_Q \sqrt{1+f_x^2+f_y^2}\,dx\,dy,$$

where Q is the normal projection of R onto the xy plane.

Solution 68.

Exercise 69. Show that

$$\mathbf{x}(u,v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha), \quad 0<u<\infty, \quad 0<v<2\pi, \quad \alpha=\text{const.},$$

is a parametrization of the cone with 2α as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve

$$\mathbf{x}(ce^{v\sin\alpha\cot\beta}, v), \quad c=\text{const.}, \ \beta=\text{const.},$$

intersects the generators of the cone ($v = \text{const.}$) under the constant angle β.

Solution 69.

Exercise 70. The coordinate curves of a parametrization $\mathbf{x}(u,v)$ constitute a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal. Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

Solution 70. The coordinate curves of a parametrization $\mathbf{x}(u,v)$ are the curves obtained by fixing one of the parameters and varying the other. Suppose we have a quadrilateral formed by the coordinate curves at points $(u_1,v_1),(u_2,v_1),(u_2,v_2),(u_1,v_2)$. Let $s(\mathbf{x}(u_1,v_1),\mathbf{x}(u_2,v_2)) \equiv s((u_1,v_1),(u_2,v_2))$ denote the arc length between two points. Then the lengths of the opposite sides are equal if and only if

$$s((u_1,v_1),(u_2,v_1)) = s((u_1,v_2),(u_2,v_2)) \implies \int_{u_1}^{u_2} \mathrm{d}u\,\sqrt{E(u,v_1)} = \int_{u_1}^{u_2} \mathrm{d}u\,\sqrt{E(u,v_2)},$$
$$s((u_1,v_1),(u_1,v_2)) = s((u_2,v_1),(u_2,v_2)) \implies \int_{v_1}^{v_2} \mathrm{d}v\,\sqrt{G(u_1,v)} = \int_{v_1}^{v_2} \mathrm{d}v\,\sqrt{G(u_2,v)}.$$

Since u_1, u_2, v_1, v_2 are arbitrary, we have

$$\sqrt{E(u,v_1)} = \sqrt{E(u,v_2)}, \quad \sqrt{G(u_1,v)} = \sqrt{G(u_2,v)}.$$

Therefore, E is independent of v and G is independent of u, giving the desired result:

$$\frac{\partial E}{\partial v} = 0, \quad \frac{\partial G}{\partial u} = 0.$$

Exercise 71 (*). Prove that whenever the coordinate curves constitute a Tchebyshef net (see Exercise 7) it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \quad F = \cos\theta, \quad G = 1,$$

where θ is the angle of the coordinate curves.

Solution 71. Following the procedure in Exercise 2.5.7, since the coordinate curves constitute a Tchebyshef net, we have $\frac{\partial E}{\partial v} = 0$ and $\frac{\partial G}{\partial u} = 0$. Thus, $E = E(u)$ and $G = G(v)$. We can define an arc length parametrization $\bar{\mathbf{x}}(\bar{u},\bar{v})$ by

$$\bar{u} = \int \sqrt{E(u)}\,\mathrm{d}u, \quad \bar{v} = \int \sqrt{G(v)}\,\mathrm{d}v,$$

Then we have

$$\bar{\mathbf{x}}_{\bar{u}} = \frac{\partial \mathbf{x}}{\partial u}\frac{\partial u}{\partial \bar{u}} = \frac{\mathbf{x}_u}{\sqrt{E(u)}}, \quad \bar{\mathbf{x}}_{\bar{v}} = \frac{\partial \mathbf{x}}{\partial v}\frac{\partial v}{\partial \bar{v}} = \frac{\mathbf{x}_v}{\sqrt{G(v)}}.$$

Thus, the coefficients of the first fundamental form in the new parametrization are

$$\begin{aligned}
\bar{E} &= \langle \bar{\mathbf{x}}_{\bar{u}}, \bar{\mathbf{x}}_{\bar{u}} \rangle = \left\langle \frac{\mathbf{x}_u}{\sqrt{E(u)}}, \frac{\mathbf{x}_u}{\sqrt{E(u)}} \right\rangle = \frac{E(u)}{E(u)} = 1, \\
\bar{F} &= \langle \bar{\mathbf{x}}_{\bar{u}}, \bar{\mathbf{x}}_{\bar{v}} \rangle = \left\langle \frac{\mathbf{x}_u}{\sqrt{E(u)}}, \frac{\mathbf{x}_v}{\sqrt{G(v)}} \right\rangle = \frac{F(u,v)}{\sqrt{E(u)G(v)}} = \cos\theta, \\
\bar{G} &= \langle \bar{\mathbf{x}}_{\bar{v}}, \bar{\mathbf{x}}_{\bar{v}} \rangle = \left\langle \frac{\mathbf{x}_v}{\sqrt{G(v)}}, \frac{\mathbf{x}_v}{\sqrt{G(v)}} \right\rangle = \frac{G(v)}{G(v)} = 1.
\end{aligned}$$

Exercise 72 (*). Show that a surface of revolution can always be parametrized so that

$$E = E(v), \quad F = 0, \quad G = 1.$$

Solution 72.

Definition 8. The coordinate curves of a parametrization are orthogonal if and only if $F(u,v) = 0$ for all (u,v). Such a parametrization is called an <u>orthogonal parametrization</u>.

Without loss of generality, let the axis of revolution be the z-axis. A surface of revolution can be parametrized as

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad u \in (0, 2\pi), \ v \in I,$$

where $f(v) > 0$ for all $v \in I$. Then

$$\begin{aligned}
\mathbf{x}_u &= (-f(v)\sin u, f(v)\cos u, 0), \\
\mathbf{x}_v &= (f'(v)\cos u, f'(v)\sin u, g'(v)).
\end{aligned}$$

The arc length parametrization of the v-curves is given by

$$\bar{v} = \int \sqrt{[f'(v)]^2 + [g'(v)]^2}\, \mathrm{d}v \implies \bar{\mathbf{x}}(u, \bar{v}) = (f(\bar{v}) \cos u, f(\bar{v}) \sin u, g(\bar{v}))\,.$$

Then, abbreviating $v(\bar{v})$ to $\bar{v}$, we have

$$\begin{aligned}
\bar{\mathbf{x}}_u &= (-f(\bar{v}) \sin u, f(\bar{v}) \cos u, 0), \\
\bar{\mathbf{x}}_{\bar{v}} &= (f'(\bar{v}) \cos u, f'(\bar{v}) \sin u, g'(\bar{v})) \frac{\mathrm{d}v}{\mathrm{d}\bar{v}} = \frac{(f'(\bar{v}) \cos u, f'(\bar{v}) \sin u, g'(\bar{v}))}{\sqrt{[f'(\bar{v})]^2 + [g'(\bar{v})]^2}}.
\end{aligned}$$

Then, the coefficients of the first fundamental form in the new parametrization are

$$\bar{E} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = f^2(\bar{v}), \quad \bar{F} = \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_{\bar{v}} \rangle = 0, \quad \bar{G} = \langle \bar{\mathbf{x}}_{\bar{v}}, \bar{\mathbf{x}}_{\bar{v}} \rangle = 1.$$

Note that if we force an arc length parametrization on the u curves instead, we would have $\bar{E} = 1$, $\bar{F} = 0$, and $\bar{G} = G(u)$. However, it is not possible to do both and still have an orthogonal parametrization.

Exercise 73. Let $P = \{(x, y, z) \in \mathbb{R}^3; z = 0\}$ be the xy plane and let $\mathbf{x} : U \to P$ be a parametrization of P given by

$$\mathbf{x}(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta),$$

where

$$U = \{(\rho, \theta) \in \mathbb{R}^2; \rho > 0, 0 < \theta < 2\pi\}.$$

Compute the coefficients of the first fundamental form of P in this parametrization.

Solution 73. We have

$$\begin{aligned}
\mathbf{x}_\rho &= (\cos \theta, \sin \theta, 0), && (16) \\
\mathbf{x}_\theta &= (-\rho \sin \theta, \rho \cos \theta, 0). && (17)
\end{aligned}$$

Thus the coefficients of the first fundamental form are

$$\begin{aligned}
E &= \langle \mathbf{x}_\rho, \mathbf{x}_\rho \rangle = \cos^2 \theta + \sin^2 \theta = 1, \\
F &= \langle \mathbf{x}_\rho, \mathbf{x}_\theta \rangle = -\rho \cos \theta \sin \theta + \rho \sin \theta \cos \theta = 0, \\
G &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle = \rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta = \rho^2.
\end{aligned}$$

Exercise 74. Let S be a surface of revolution and C its generating curve (cf. Example 4, Sec. 2-3). Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

a. (*Pappus' Theorem.*) Show that the area of S is

$$2\pi \int_0^l \rho(s)\, ds,$$

where l is the length of C.

b. Apply part (a) to compute the area of a torus of revolution.

Solution 74.

Exercise 75. Show that the area of a regular tube of radius r around a curve α (cf. Exercise 10, Sec. 2-4) is $2\pi r$ times the length of α.

Solution 75. From Exercise 2.4.10, the parametrization of a tubular surface of radius r around a curve α is given by

$$\mathbf{x}(s,\theta) = \alpha(s) + r\left(n(s)\cos\theta + b(s)\sin\theta\right), \quad r \neq 0, s \in I.$$

Then

$$\begin{aligned}
\mathbf{x}_s &= \alpha'(s) + r\left(n'(s)\cos\theta + b'(s)\sin\theta\right)\\
&= t(s) + r\left(-k(s)t(s)\cos\theta - \tau(s)b(s)\cos\theta + \tau(s)n(s)\sin\theta\right)\\
&= (1 - rk\cos\theta)t + r\tau\sin\theta n - r\tau\cos\theta b,\\
\mathbf{x}_\theta &= r\left(-n(s)\sin\theta + b(s)\cos\theta\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_s \wedge \mathbf{x}_\theta &= ((1 - rk\cos\theta)t + r\tau\sin\theta n - r\tau\cos\theta b) \wedge r\left(-n(s)\sin\theta + b(s)\cos\theta\right)\\
&= -r(1 - rk(s)\cos\theta)(\cos\theta\, n(s) + \sin\theta\, b(s)).
\end{aligned}$$

By definition 7, the area of the tube is given by

$$A = \iint \mathrm{d}\theta\,\mathrm{d}s\,|\mathbf{x}_s \wedge \mathbf{x}_\theta| = \int_0^{2\pi}\mathrm{d}\theta\int_{s_1}^{s_2}\mathrm{d}s\, r|1 - rk(s)\cos\theta|.$$

Since $k(s)r \leq 1$ for all $s \in I$, we have

$$A = \int_0^{2\pi}\mathrm{d}\theta\int_{s_1}^{s_2}\mathrm{d}s\, r(1 - rk(s)\cos\theta) = 2\pi r\int_{s_1}^{s_2}\mathrm{d}s = 2\pi r\,\ell(\alpha).$$

Remark. Let's formalize the problem in the following way: Let $\alpha : [0,\ell] \to \mathbb{R}^3$ be a curve parametrized by arc length with $k(s) \neq 0$. Suppose α has no self-intersections, $\alpha(0) = \alpha(\ell)$ and it induces a smooth map from S^1 to $\mathbb{R}^3$ (i.e. α is a smooth simple closed curve). Let $r > 0$ and $\varphi : [0,\ell] \times [0,2\pi] \to \mathbb{R}^3$ is given by:

$$\varphi(s,v) = \alpha(s) + r(n(s))\cos v + b(s)\sin v$$

Then image $T = \operatorname{Im}\varphi$ is called the tube of radius r around α. For r sufficiently small, T is a surface. Prove that $A(T) = 2\pi r\ell$.

Exercise 76. (*Generalized Helicoids.*) A natural generalization of both surfaces of revolution and helicoids is obtained as follows. Let a regular plane curve C, which does not meet an axis E in the plane, be displaced in a rigid screw motion about E, that is, so that each point of C describes a helix (or circle) with E as axis. The set S generated by the displacement of C is called a *generalized helicoid* with axis E and generator C. If the screw motion is a pure rotation about E, S is a surface of revolution; if C is a straight line perpendicular to E, S is (a piece of) the standard helicoid (cf. Example 3).

Choose the coordinate axes so that E is the z axis and C lies in the yz plane. Prove that

a. If $(f(s), g(s))$ is a parametrization of C by arc length s, $a < s < b$, $f(s) > 0$, then $\mathbf{x} : U \to S$, where
$$U = \{(s,u) \in \mathbb{R}^2; a < s < b, 0 < u < 2\pi\}$$
and
$$\mathbf{x}(s,u) = (f(s)\cos u, f(s)\sin u, g(s) + cu), \quad c = \text{const.},$$
is a parametrization of S. Conclude that S is a regular surface.

b. The coordinate lines of the above parametrization are orthogonal (i.e., $F = 0$) if and only if $\mathbf{x}(U)$ is either a surface of revolution or (a piece of) the standard helicoid.

Solution 76.

Exercise 77 (*Gradient on Surfaces.*). The gradient of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\operatorname{grad} f : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\operatorname{grad} f(p) \in T_p(S) \subset \mathbb{R}^3$ such that

$$(\operatorname{grad} f(p), v)_p = df_p(v) \quad \text{for all } v \in T_p(S).$$

Show that

a. If E, F, G are the coefficients of the first fundamental form in a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, then $\operatorname{grad} f$ on $\mathbf{x}(U)$ is given by

$$\operatorname{grad} f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v.$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y,

$$\operatorname{grad} f = f_x e_1 + f_y e_2,$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ (thus, the definition agrees with the usual definition of gradient in the plane).

b. If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in $T_p(S)$, then $df_p(v)$ is maximum if and only if $v = \operatorname{grad} f / |\operatorname{grad} f|$ (thus, $\operatorname{grad} f(p)$ gives the direction of maximum variation of f at p).

c. If $\operatorname{grad} f \neq 0$ at all points of the level curve $C = \{q \in S; f(q) = \text{const.}\}$, then C is a regular curve on S and $\operatorname{grad} f$ is normal to C at all points of C.

Solution 77.

a.

Exercise 78 (*Orthogonal Families of Curves.*).

a. Let E, F, G be the coefficients of the first fundamental form of a regular surface S in the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. Let $\varphi(u, v) = \text{const.}$ and $\psi(u, v) = \text{const.}$ be two families of regular curves on $\mathbf{x}(U) \subset S$ (cf. Exercise 28, Sec. 2-4). Prove that these two families are orthogonal (i.e., whenever two curves of distinct families meet, their tangent lines are orthogonal) if and only if

$$E\varphi_u\psi_v - F(\varphi_u\psi_u + \varphi_v\psi_v) + G\varphi_v\psi_u = 0.$$

b. Apply part (a) to show that on the coordinate neighborhood $\mathbf{x}(U)$ of the helicoid of Example 3, the two families of regular curves

$$v \cos u = \text{const.}, \quad v \neq 0,$$

$$(v^2 + a^2)\sin^2 u = \text{const.}, \quad v \neq 0, \quad u \neq \pi,$$

are orthogonal.

Solution 78.

a. Suppose $\varphi(u, v) = c_1$ and $\psi(u, v) = c_2$ are two families of regular curves on $\mathbf{x}(U) \subset S$. The curves satisfy

$$\phi_u u' + \phi_v v' = 0, \quad \psi_u u' + \psi_v v' = 0.$$

So we can choose the tangent vectors of the two families of curves to be

$$t(\varphi) = -\varphi_v \mathbf{x}_u + \varphi_u \mathbf{x}_v, \quad t(\psi) = -\psi_v \mathbf{x}_u + \psi_u \mathbf{x}_v.$$

The two families are orthogonal if and only if $\langle t(\varphi), t(\psi) \rangle = 0$, which is equivalent to

$$\begin{aligned}\langle t(\varphi), t(\psi) \rangle &= \langle -\varphi_v \mathbf{x}_u + \varphi_u \mathbf{x}_v, -\psi_v \mathbf{x}_u + \psi_u \mathbf{x}_v \rangle \\ &= \varphi_v \psi_v \langle \mathbf{x}_u, \mathbf{x}_u \rangle - \varphi_v \psi_u \langle \mathbf{x}_u, \mathbf{x}_v \rangle - \varphi_u \psi_v \langle \mathbf{x}_v, \mathbf{x}_u \rangle + \varphi_u \psi_u \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\ &= E\varphi_v \psi_v - F(\varphi_u \psi_v + \varphi_v \psi_u) + G\varphi_u \psi_u = 0.\end{aligned}$$

b. From Example 3, the helicoid is given by the parametrization $\mathbf{x}(u, v) = (v \cos u, v \sin u, au)$, with the coefficients of the first fundamental form being

$$E(u, v) = v^2 + a^2, \quad F(u, v) = 0, \quad G(u, v) = 1.$$

Let $\phi(u, v) = v \cos u$, $\psi(u, v) = (v^2 + a^2) \sin^2 u$. Then

$$\begin{aligned}\phi_u &= -v \sin u, & \phi_v &= \cos u, \\ \psi_u &= 2(v^2 + a^2) \sin u \cos u, & \psi_v &= 2v \sin^2 u.\end{aligned}$$

Substituting these into equation (**??**) in part (a), we have

$$(v^2 + a^2) \cos u(2v \sin^2 u) - 0 + 1(-v \sin u)(2(v^2 + a^2) \sin u \cos u) = 0.$$

Therefore, the two families of regular curves are orthogonal.