2025 Fall Introduction to Geometry

Solutions to Exercises in Do Carmo

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Definition 1 (isometry)**.** A diffeomorphism $\varphi : S \to \overline{S}$ is an <u>isometry</u> if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have

$$\langle w_1, w_2 \rangle_p = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle_{\varphi(p)}.$$

The surfaces S and $\overline{S}$ are then said to be <u>isometric</u>.

Remark. An isometry is a diffeomorphism that preserves the first fundamental form.

Proposition 1 (Do Carmo Proposition 4.2.1)**.** Assume the existence of parametrizations $\mathbf{x} : U \to S$ and $\overline{\mathbf{x}} : U \to \overline{S}$ such that $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$ in U. Then $\overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{S}$ is a local isometry.

Exercise 4.2.5. Let $\alpha_1 : I \to \mathbb{R}^3$, $\alpha_2 : I \to \mathbb{R}^3$ be regular parametrized curves, where the parameter is the arc length. Assume that the curvatures k_1 of α_1 and k_2 of α_2 satisfy

$$k_1(s) = k_2(s) \neq 0, \quad s \in I.$$

Let

$$\mathbf{x}_1(s, v) = \alpha_1(s) + v\alpha_1'(s), \qquad \mathbf{x}_2(s, v) = \alpha_2(s) + v\alpha_2'(s)$$

be their (regular) tangent surfaces (cf. Example 5, Sec. 2-3) and let V be a neighborhood of (s_0, v_0) such that $\mathbf{x}_1(V) \subset \mathbb{R}^3$, $\mathbf{x}_2(V) \subset \mathbb{R}^3$ are regular surfaces (cf. Prop. 2, Sec. 2-3). Prove that

$$\mathbf{x}_1 \circ \mathbf{x}_2^{-1} : \mathbf{x}_2(V) \longrightarrow \mathbf{x}_1(V)$$

is an isometry.

Solution 4.2.5. To show that $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is an isometry, we need to show that it is a diffeomorphism and preserves the first fundamental form. From Example 2.3.5, the tangent surface of a regular curve α is a regular surface, since for all $(t, v) \subseteq U = \{(t, v) \in I \times \mathbb{R} \mid v \neq 0\}$, we have

$$k(s) = \frac{|\alpha'(s) \wedge \alpha''(s)|}{|\alpha'(s)|^3} \neq 0 \implies \frac{\partial \mathbf{x}}{\partial s} \wedge \frac{\partial \mathbf{x}}{\partial v} = v\alpha''(s) \wedge \alpha'(s) \neq 0.$$

Thus, both $\mathbf{x}_1$ and $\mathbf{x}_2$ are regular parametrizations, and hence homeomorphisms on a small neighborhood $V \subseteq \mathbb{R}^3$. Since $\mathbf{x}$ is differentiable and $\mathrm{d}\mathbf{x}_i$ has full rank, $\mathbf{x}_i^{-1}$ is differentiable for $i = 1, 2$ by the Inverse Function Theorem. Therefore, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is a diffeomorphism. In the Frenet frames of α_i, $i = 1, 2$, we have $\mathbf{x}_i(s, v) = \alpha_i(s) + v\alpha'(s)$, and

$$\mathbf{x}_{i,s} = \alpha'(s) + v\alpha''(s) = T_i(s) + vk_i(s)N_i(s), \quad \mathbf{x}_{i,v} = \alpha'(s) = T_i(s).$$

The first fundamental form coefficients are computed to be

$$E_i = \langle \mathbf{x}_{i,s}, \mathbf{x}_{i,s} \rangle = 1 + v^2 k_i^2(s), \quad F_i = \langle \mathbf{x}_{i,s}, \mathbf{x}_{i,v} \rangle = 1, \quad G_i = \langle \mathbf{x}_{i,v}, \mathbf{x}_{i,v} \rangle = 1.$$

Since $k_1(s) = k_2(s)$ for all $s \in I$, we have $E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$. By Proposition 4.2.1, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is a local isometry. Since $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is also a diffeomorphism, $\mathbf{x}_1 \circ \mathbf{x}_2^{-1}$ is an isometry.

Exercise 4.2.6*. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with $k(t) \neq 0$, $t \in I$. Let $\mathbf{x}(t, v)$ be its tangent surface. Prove that, for each $(t_0, v_0) \in I \times (\mathbb{R} - \{0\})$, there exists a neighborhood V of (t_0, v_0) such that $\mathbf{x}(V)$ is isometric to an open set of the plane (thus, tangent surfaces are locally isometric to planes).

Solution 4.2.6. We will construct the desired local isometry. Fir reparametrize by arc length to get $\alpha(s)$, and define $\mathbf{x}(s, v) = \alpha(s) + v\alpha'(s)$. Let $k(s)$ be the curvature of $\alpha(s)$. As in a previous exercise, let

$$\theta(s) = \int_{s_0}^{s} \mathrm{d}u \, k(u), \quad s_0 \in I$$

be the angle function, and define a plane curve $\beta(s)$ by

$$\beta(s) = \left(\int_{s_0}^{s} \mathrm{d}u \, \cos\theta(u), \int_{s_0}^{s} \mathrm{d}u \, \sin\theta(u), 0 \right),$$

$$\beta'(s) = (\cos\theta(s), \sin\theta(s), 0) \implies |\beta'(s)| = 1,$$

$$\beta''(s) = \theta'(s) \left(-\sin\theta(s), \cos\theta(s), 0\right) = k(s) \left(-\sin\theta(s), \cos\theta(s), 0\right).$$

Then, the curvature of $\beta(s)$ is exactly $k(s)$, and hence $\beta(s)$ is a unit-speed curve with the same curvature as α. Since both β and β' lie in the plane $z = 0$, the image of the tangent surface $\overline{\mathbf{x}}(s, v) = \beta(s) + v\beta'(s)$ is an open subset of the xy-plane. For $\mathbf{x}$ and $\overline{\mathbf{x}}$, we have

$$\mathbf{x}_s = T(s) + vk(s)N(s), \quad \mathbf{x}_v = T(s),$$

$$\overline{\mathbf{x}}_s = \overline{T}(s) + vk(s)\overline{N}(s), \quad \overline{\mathbf{x}}_v = \overline{T}(s),$$

where $T, N, \overline{T}, \overline{N}$ are the tangent vector and normal vector of $\mathbf{x}$ and $\overline{\mathbf{x}}$, respectively. The first fundamental form coefficients of $\mathbf{x}$ and $\overline{\mathbf{x}}$ are, respectively,

$$E = 1 + v^2 k^2(s), \quad F = 1, \quad G = 1,$$

$$\overline{E} = 1 + v^2 k^2(s), \quad \overline{F} = 1, \quad \overline{G} = 1.$$

Since the coefficients agree, by Proposition 4.2.1, the map $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ is a local isometry from $\mathbf{x}(V)$ to an open set of the plane for some neighborhood V of (s_0, v_0). Therefore, the tangent surface is locally isometric to an open set of the plane.

Exercise 4.2.7. Let V and W be n-dimensional vector spaces with inner products denoted by $\langle\, , \,\rangle$ and let $F : V \to W$ be a linear map. Prove that the following conditions are equivalent:

a. $\langle F(v_1), F(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$.

b. $|F(v)| = |v|$ for all $v \in V$.

c. If $\{v_1, \ldots, v_n\}$ is an orthonormal basis in V, then $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.

d. There exists an orthonormal basis $\{v_1, \ldots, v_n\}$ in V such that $\{F(v_1), \ldots, F(v_n)\}$ is an orthonormal basis in W.

If any of these conditions is satisfied, F is called a linear isometry of V into W. (When $W = V$, a linear isometry is often called an orthogonal transformation.)

Solution 4.2.7.

- **a. $\Longrightarrow$ b.** Suppose $\langle F(v_1), F(v_2)\rangle = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in V$. Then for all $v \in V$,

$$|v| = \sqrt{\langle v, v\rangle} = \sqrt{\langle F(v), F(v)\rangle} = |F(v)|.$$

- **b. $\Longrightarrow$ c.** Suppose $|F(v)| = |v|$ for all $v \in V$. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of V. Then, for all $i, j = 1, \dots, n$, since the inner product is induced by a norm $|\cdot|$, we have

$$\begin{aligned}\langle F(v_i), F(v_j)\rangle &= \frac{1}{2}\left(|F(v_i) + F(v_j)|^2 - |F(v_i)|^2 - |F(v_j)|^2\right)\\ &= \frac{1}{2}\left(|v_i + v_j|^2 - |v_i|^2 - |v_j|^2\right) = \langle v_i, v_j\rangle = \delta_{ij}.\end{aligned}$$

  Thus, $\{F(v_1), \dots, F(v_n)\}$ is an orthonormal set in W. Since F is linear, $\{F(v_1), \dots, F(v_n)\}$ spans $\mathrm{Im}(F)$. Since $\dim(\mathrm{Im}(F)) \leq n$, we have $\dim(\mathrm{Im}(F)) = n$, and hence $\{F(v_1), \dots, F(v_n)\}$ is an orthonormal basis of W.

- **c. $\Longrightarrow$ d.** Since V is finite-dimensional, just pick any orthonormal basis of V.

- **d. $\Longrightarrow$ a.** Suppose there exists an orthonormal basis $\{v_1, \dots, v_n\}$ of V such that $\{F(v_1), \dots, F(v_n)\}$ is an orthonormal basis of W. For all $v_1, v_2 \in V$, we can write

$$v_1 = \sum_{i=1}^n a_i v_i, \quad v_2 = \sum_{j=1}^n b_j v_j,$$

  where $a_i, b_j \in \mathbb{R}$. Then,

$$\begin{aligned}\langle F(v_1), F(v_2)\rangle &= \left\langle F\left(\sum_{i=1}^n a_i v_i\right), F\left(\sum_{j=1}^n b_j v_j\right)\right\rangle\\ &= \left\langle \sum_{i=1}^n a_i F(v_i), \sum_{j=1}^n b_j F(v_j)\right\rangle\\ &= \sum_{i=1}^n \sum_{j=1}^n a_i b_j \langle F(v_i), F(v_j)\rangle\\ &= \sum_{i=1}^n \sum_{j=1}^n a_i b_j \delta_{ij} = \sum_{i=1}^n a_i b_i = \left\langle \sum_{i=1}^n a_i v_i, \sum_{j=1}^n b_j v_j\right\rangle = \langle v_1, v_2\rangle.\end{aligned}$$

Exercise 4.2.8*. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that

$$|G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in \mathbb{R}^3$$

(that is, G is a distance-preserving map). Prove that there exists $p_0 \in \mathbb{R}^3$ and a linear isometry (cf. Exercise 7) F of the vector space $\mathbb{R}^3$ such that

$$G(p) = F(p) + p_0 \quad \text{for all } p \in \mathbb{R}^3.$$

Solution 4.2.8. Let $p_0 = G(0)$, and let $F(p) = G(p) - p_0$. Then, for all $p, q \in \mathbb{R}^3$, we have

$$|F(p) - F(q)| = |G(p) - G(q)| = |p - q|, \quad F(0) = G(0) - p_0 = 0.$$

Hence F is a distance-preserving map that fixes the origin. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, and $v_i = F(e_i)$ for $i = 1, 2, 3$. Since F is distance-preserving, we have

$$|v_i|^2 = |F(e_i) - F(0)|^2 = |e_i - 0|^2 = 1, \quad |v_i - v_j|^2 = |F(e_i) - F(e_j)|^2 = |e_i - e_j|^2 = 2,$$

squaring both sides gives

$$\langle v_i, v_j\rangle = 0 \text{ for } i \neq j \implies \{v_1, v_2, v_3\} \text{ is an orthonormal basis for } \mathbb{R}^3.$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(e_i) = v_i$ for $i = 1, 2, 3$. Then L is linear by construction, and $L(e_i) = v_i = F(e_i)$, $i = 1, 2, 3$. For any $p \in \mathbb{R}^3$, since $L(0) = 0$, by the distance-preserving property of F, we have $|F(p)| = |p| = |L(p)|$. Then, for all $i = 1, 2, 3$, we have

$$|F(p) - F(e_i)| = |p - e_i| = |L(p) - L(e_i)| .$$

Squaring both sides, then using $|F(p)| = |L(p)|$ and $F(e_i) = L(e_i)$, we have $\langle F(p)-L(p), F(e_i)\rangle = 0$. Hence, $F = L$, and F is linear. By Exercise 4.3.7, F is a linear isometry. Therefore, there exists a linear isometry F such that $G(p) = F(p) + p_0$ for all $p \in \mathbb{R}^3$.

Exercise 4.2.9. Let S_1, S_2, and S_3 be regular surfaces. Prove that

- **a.** If $\varphi : S_1 \to S_2$ is an isometry, then $\varphi^{-1} : S_2 \to S_1$ is also an isometry.
- **b.** If $\varphi : S_1 \to S_2$, $\psi : S_2 \to S_3$ are isometries, then $\psi \circ \varphi : S_1 \to S_3$ is an isometry.

This implies that the isometries of a regular surface S constitute in a natural way a group, called the group of isometries of S.

Solution 4.2.9.

- **a.** Since φ is an isometry, for all $p \in S_1$ and all pairs $w_1, w_2 \in T_p(S_1)$ we have

  $$\langle w_1, w_2\rangle_p = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2)\rangle_{\varphi(p)}.$$

  Let $q = \varphi(p) \in S_2$ and $u_1, u_2 \in T_q(S_2)$. Since φ is a diffeomorphism, $\mathrm{d}\varphi$ is injective. Since the differential $\mathrm{d}\varphi$ is a linear transformation between finite-dimensional spaces, it is also surjective. Thus, there exist $w_1, w_2 \in T_p(S_1)$ such that $\mathrm{d}\varphi_p(w_i) = u_i$ for $i = 1, 2$. Thus,

  $$\langle \mathrm{d}\varphi_p^{-1}(u_1), \mathrm{d}\varphi_p^{-1}(u_2)\rangle_q = \langle w_1, w_2\rangle_p = \langle u_1, u_2\rangle_{\varphi(p)}.$$

  Therefore, φ^{-1} is an isometry.
- **b.** Suppose $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are isometries. Since diffeomorphism between regular surfaces is an equivalence relation (by previous exercise), $\psi \circ \varphi$ is a diffeomorphism. For all $p \in S_1$ and all pairs $w_1, w_2 \in T_p(S_1)$, we have

  $$\begin{aligned}\langle w_1, w_2\rangle_p &= \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2)\rangle_{\varphi(p)} \\ &= \langle \mathrm{d}\psi_{\varphi(p)}(\mathrm{d}\varphi_p(w_1)), \mathrm{d}\psi_{\varphi(p)}(\mathrm{d}\varphi_p(w_2))\rangle_{\psi(\varphi(p))} \\ &= \langle \mathrm{d}(\psi \circ \varphi)_p(w_1), \mathrm{d}(\psi \circ \varphi)_p(w_2)\rangle_{(\psi\circ\varphi)(p)},\end{aligned}$$

  where the chain rule is used in the last equality. Therefore, $\psi \circ \varphi$ is an isometry.

Remark. Since function composition is associative and the identity map $\mathrm{id} : S_1 \to S_1$ is an isometry, by **a.** and **b.**, the set of isometries on S forms a group.

3 Chapter 4.3

Theorem 2 (Theorema Egregium)**.** The Gaussian curvature K of a regular, orientable, and oriented surface S is invariant under local isometries. Explicitly, for a parametrization $\mathbf{x}(u,v)$ in the orientation of S, we have

$$-EK = \left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2.$$

Proof. This is adapted from Do Carmo Curve and Surfaces. Define the Christoffel symbols of S in the parametrization $\mathbf{x}(u,v)$ by

$$\begin{cases} \mathbf{x}_{uu} = \Gamma_{11}^1\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_v + eN, \\ \mathbf{x}_{uv} = \Gamma_{12}^1\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v + fN, \\ \mathbf{x}_{vu} = \Gamma_{21}^1\mathbf{x}_u + \Gamma_{21}^2\mathbf{x}_v + fN, \\ \mathbf{x}_{vv} = \Gamma_{22}^1\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_v + gN, \\ N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\ N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v. \end{cases}$$

Take inner products with $\mathbf{x}_u$ and $\mathbf{x}_v$, we have

$$\begin{cases} \Gamma_{11}^1E + \Gamma_{11}^2F = \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle = \dfrac{E_u}{2}, \\ \Gamma_{11}^1F + \Gamma_{11}^2G = \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle = F_u - \dfrac{1}{2}E_v. \end{cases}$$
$$\begin{cases} \Gamma_{12}^1E + \Gamma_{12}^2F = \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = \dfrac{1}{2}E_v, \\ \Gamma_{12}^1F + \Gamma_{12}^2G = \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = \dfrac{1}{2}G_u. \end{cases}$$
$$\begin{cases} \Gamma_{22}^1E + \Gamma_{22}^2F = \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle = F_v - \dfrac{1}{2}G_u, \\ \Gamma_{22}^1F + \Gamma_{22}^2G = \langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle = \dfrac{1}{2}G_v. \end{cases}$$

By smoothness, we have $\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = 0$, so expressing the equation as $A_1\mathbf{x}_u + B_1\mathbf{x}_v + C_1N = 0$ gives $A_1 = B_1 = C_1 = 0$. ☐

Corollary 3. For each pair, the determinant of the coefficient matrix is $EG - F^2 \neq 0$, so we can solve for the Christoffel symbols explicitly.

$$\Gamma_{11}^1 =$$

Corollary 4. By Theorema Egregium, the Gaussian curvature K can be computed entirely in terms of the first fundamental form coefficients E, F, G and their derivatives. This given explicitly by

Theorem 5 (Mainardi-Codazzi equations)**.** With the same notation as above, we have

$$e_v - f_u = e\Gamma_{12}^1 + f\left(\Gamma_{12}^2 - \Gamma_{11}^1\right) - g\Gamma_{11}^2, \quad f_v - g_u = e\Gamma_{22}^1 + f\left(\Gamma_{22}^2 - \Gamma_{12}^1\right) - g\Gamma_{12}^2.$$

Remark. The Gauss equation and the Mainardi-Codazzi equations are known as the compatibility equations of the theory of surfaces.

Theorem 6 (Bonnet)**.** Let E, F, G, e, f, g be differentiable functions defined on an open set $V \subseteq \mathbb{R}^2$, with $E, G > 0$. Suppose that these functions satisfy the compatibility equations, and $\det g = EG - F^2 > 0$. Then, for each point $p \in V$, there exists a neighborhood $U \subseteq V$ and a regular diffeomorphism $\mathbf{x} : U \to \mathbb{R}^3$ such that the coefficients of the first fundamental form of $\mathbf{x}$ are E, F, G, and those of the second fundamental form are e, f, g. Moreover, if U is connected and $\overline{\mathbf{x}} : U \to \overline{\mathbf{x}}(U)$ is another diffeomorphism satisfying the same conditions, then $\overline{\mathbf{x}} = T \circ \rho \circ \mathbf{x}$ for some translation T and rotation ρ.

Lemma 1 (Gaussian curvature)**.** The Gaussian curvature K of a regular surface is given by

$$K = \frac{eg - f^2}{EG - F^2}.$$

Proof. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a regular surface S. Then, we have

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ e &= \langle \mathbf{x}_{uu}, N \rangle, \quad f = \langle \mathbf{x}_{uv}, N \rangle, \quad g = \langle \mathbf{x}_{vv}, N \rangle, \end{aligned}$$

where N is the unit normal. In the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, the first and second fundamental forms are

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The shape operator $S : T_pS \to T_pS$ is defined by $S(v) = -\mathrm{d}N_v$, with the principal curvatures k_1, k_2 being its eigenvalues. It has been shown that $S = g^{-1}A$, so

$$K = \det S = \det\left(g^{-1}A\right) = \frac{\det A}{\det g} = \frac{eg - f^2}{EG - F^2}.$$

□

Exercise 4.3.1. Show that if $\mathbf{x}$ is an orthogonal parametrization, that is, $F = 0$, then

$$K = -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}.$$

Solution 4.3.1. From the definition of the Christoffel symbols, we have

$$\begin{aligned} \mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L_1 N, \\ \mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + L_2 N, \\ \mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + L_3 N, \end{aligned}$$

we can compute the relations satisfied by the Christoffel symbols by taking inner product with $\mathbf{x}_u$ and $\mathbf{x}_v$ for each of the three equations above. Then, we get

$$\begin{aligned} \Gamma^1_{11}E + \Gamma^2_{11}F &= \frac{E_u}{2}, \quad \Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \frac{E_v}{2}, \\ \Gamma^1_{12}E + \Gamma^2_{12}F &= \frac{E_v}{2}, \quad \Gamma^1_{12}F + \Gamma^2_{12}G = \frac{G_u}{2}, \\ \Gamma^1_{22}E + \Gamma^2_{22}F &= F_v - \frac{G_u}{2}, \quad \Gamma^1_{22}F + \Gamma^2_{22}G = \frac{G_v}{2}. \end{aligned}$$

Since $F = 0$ and $\Gamma^i_{jk} = \Gamma^i_{kj}$, we have

$$\begin{aligned} \Gamma^1_{11} &= \frac{E_u}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E}, \\ \Gamma^2_{12} &= \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2E}, \quad \Gamma^2_{22} = \frac{G_v}{2G}. \end{aligned}$$

and taking inner product with N gives $L_1 = e$, $L_2 = f$, $L_3 = g$. Thus, we have

$$\begin{aligned}
\mathbf{x}_{uu} &= \frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN, \\
\mathbf{x}_{uv} &= \frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN, \\
\mathbf{x}_{vv} &= -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + gN.
\end{aligned}$$

Next, use equation (1) in Section 4.3 to get

$$\begin{aligned}
N_u &= \frac{fF - eG}{EG - F^2}\mathbf{x}_u + \frac{eF - fE}{EG - F^2}\mathbf{x}_v = -\frac{e}{E}\mathbf{x}_u - \frac{f}{G}\mathbf{x}_v, \\
N_v &= \frac{gF - fG}{EG - F^2}\mathbf{x}_u + \frac{fF - gE}{EG - F^2}\mathbf{x}_v = -\frac{f}{E}\mathbf{x}_u - \frac{g}{G}\mathbf{x}_v.
\end{aligned}$$

Since the parametrization is continuously differentiable, the partial derivatives commute, and we have $\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = 0$. First, let's compute the following partial derivatives:

$$\left(\frac{E_v}{2G}\right)_v = \frac{E_{vv}}{2G} - \frac{E_v G_v}{2G^2}, \quad \left(\frac{G_u}{2G}\right)_u = \frac{G_{uu}}{2G} - \frac{(G_u)^2}{2G^2}.$$

Next, we will compute $\mathbf{x}_{uuv}$:

$$\begin{aligned}
\mathbf{x}_{uuv} = (x_{uu})_v &= \left(\frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN\right)_v \\
&= \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\mathbf{x}_{uv} - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v - \frac{E_v}{2G}\mathbf{x}_{vv} + e_v N + eN_v \\
&= \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\left[\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right] - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v \\
&\quad - \frac{E_v}{2G}\left[-\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + gN\right] + e_v N + e\left(-\frac{f}{E}\mathbf{x}_u - \frac{g}{E}\mathbf{x}_v\right) \\
&= \left[\left(\frac{E_u}{2E}\right)_v + \frac{E_u E_v}{4E^2} + \frac{E_v G_u}{4EG} - \frac{ef}{E}\right]\mathbf{x}_u + \left[-\left(\frac{E_v}{2G}\right)_v + \frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} - \frac{eg}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{E_u f}{2E} - \frac{E_v g}{2G} + e_v\right] N.
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
\mathbf{x}_{uvu} = (x_{uv})_u &= \left(\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right)_u \\
&= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\mathbf{x}_{uu} + \left(\frac{G_u}{2G}\right)_u \mathbf{x}_v + \frac{G_u}{2G}\mathbf{x}_{uv} + f_u N + fN_u \\
&= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\left[\frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + eN\right] \\
&\quad + \left(\frac{G_u}{2G}\right)_u \mathbf{x}_v + \frac{G_u}{2G}\left[\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + fN\right] + f_u N + f\left(-\frac{e}{G}\mathbf{x}_u - \frac{f}{G}\mathbf{x}_v\right) \\
&= \left[\left(\frac{E_v}{2E}\right)_u + \frac{E_u E_v}{4E^2} + \frac{E_v G_u}{4EG} - \frac{ef}{E}\right]\mathbf{x}_u + \left[\left(\frac{G_u}{2G}\right)_u - \frac{(E_v)^2}{4EG} + \frac{(G_u)^2}{4G^2} - \frac{f^2}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{E_v e}{2E} + \frac{G_u f}{2G} + f_u\right] N.
\end{aligned}$$

Combining the two results above, we have

$$\begin{aligned}
\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = &\left[\left(\frac{E_u}{2E}\right)_v - \left(\frac{E_v}{2E}\right)_u\right]\mathbf{x}_u + \left[\frac{E_u f - E_v e}{2E} - \frac{E_v g - G_u f}{2G} + e_v - f_u\right] N \\
&+ \left[\frac{E_u G_u + (E_v)^2}{4EG} - \frac{E_v G_v + (G_u)^2}{4G^2} - \frac{eg - f^2}{G} - \left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u\right]\mathbf{x}_v = 0.
\end{aligned}$$

Since $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ is an orthonormal basis, each coefficient is equal to zero. Set the coefficient of $\mathbf{x}_v$ to zero and recall the formula for the Gaussian curvature:

$$\begin{aligned}
K &= \frac{eg-f^2}{EG-F^2} = \frac{eg-f^2}{EG} \\
&= \frac{E_uG_u+(E_v)^2}{4E^2G} - \frac{E_vG_v+(G_u)^2}{4EG^2} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u \\
&= \frac{E_uG_u}{4E^2G} + \frac{(E_v)^2}{4E^2G} - \frac{E_vG_v}{4EG^2} - \frac{(G_u)^2}{4EG^2} - \frac{E_{vv}}{2EG} + \frac{E_vG_v}{2EG^2} - \frac{G_{uu}}{2EG} + \frac{(G_u)^2}{2EG^2} \\
&= -\frac{1}{2\sqrt{EG}}\left[\frac{G_{uu}}{\sqrt{EG}} - \frac{E_uG_u}{2E\sqrt{EG}} - \frac{(G_u)^2}{2G\sqrt{EG}} + \frac{E_{vv}}{\sqrt{EG}} - \frac{(E_v)^2}{2E\sqrt{EG}} - \frac{E_vG_v}{2G\sqrt{EG}}\right] \\
&= -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.
\end{aligned}$$

Remark. The above formual for the Gaussian curvature of orthogonal parametrizations is known as the Brioschi formula.

Exercise 4.3.2. Show that if $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then

$$K = -\frac{1}{2\lambda}\,\Delta(\log\lambda),$$

where $\Delta\varphi$ denotes the Laplacian $(\partial^2\varphi/\partial u^2) + (\partial^2\varphi/\partial v^2)$ of the function φ. Conclude that when

$$E = G = (u^2+v^2+c)^{-2} \quad \text{and} \quad F = 0,$$

then $K = \text{const.} = 4c$.

Solution 4.3.2. Suppose $\mathbf{x}$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$. Then we have

$$\begin{aligned}
E_v = \lambda_v, &\quad G_u = \lambda_u, \\
E_{vv} = \lambda_{vv}, &\quad G_{uu} = \lambda_{uu}.
\end{aligned}$$

From the proof of Exercise 4.3.1, since an isothermal parametrization is orthogonal, we have

$$\begin{aligned}
K &= -\frac{1}{2\sqrt{EG}}\left[\frac{G_{uu}}{\sqrt{EG}} - \frac{E_uG_u}{2E\sqrt{EG}} - \frac{(G_u)^2}{2G\sqrt{EG}} + \frac{E_{vv}}{\sqrt{EG}} - \frac{(E_v)^2}{2E\sqrt{EG}} - \frac{E_vG_v}{2G\sqrt{EG}}\right] \\
&= -\frac{1}{2\lambda}\left[\frac{\lambda_{uu}}{\lambda} - \frac{\lambda_u^2}{2\lambda^2} - \frac{\lambda_u^2}{2\lambda^2} + \frac{\lambda_{vv}}{\lambda} - \frac{\lambda_v^2}{2\lambda^2} - \frac{\lambda_v^2}{2\lambda^2}\right] \\
&= -\frac{1}{2\lambda}\left[\frac{\lambda_{uu}+\lambda_{vv}}{\lambda} - \frac{\lambda_u^2+\lambda_v^2}{\lambda^2}\right] = -\frac{1}{2\lambda}\Delta(\log\lambda),
\end{aligned}$$

since

$$\Delta(\log\lambda) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)(\log\lambda) = \frac{\partial}{\partial u}\left(\frac{\lambda_u}{\lambda}\right) + \frac{\partial}{\partial v}\left(\frac{\lambda_v}{\lambda}\right) = \frac{\lambda_{uu}+\lambda_{vv}}{\lambda} - \frac{\lambda_u^2+\lambda_v^2}{\lambda^2}.$$

Let $E = G = (u^2+v^2+c)^{-2}$ and $F = 0$, then we have $\lambda(u, v) = (u^2+v^2+c)^{-2}$. Then,

$$\begin{aligned}
\frac{\partial}{\partial u}(\log\lambda) &= -2\frac{\partial}{\partial u}\log\left(u^2+v^2+c\right) = -\frac{4u}{u^2+v^2+c}, \\
\frac{\partial^2}{\partial u^2}(\log\lambda) &= -4\frac{\partial}{\partial u}\left(\frac{u}{u^2+v^2+c}\right) = -4\,\frac{(-u^2+v^2+c)}{(u^2+v^2+c)^2}, \\
\frac{\partial}{\partial v}(\log\lambda) &= -2\frac{\partial}{\partial v}\log\left(u^2+v^2+c\right) = -\frac{4v}{u^2+v^2+c}, \\
\frac{\partial^2}{\partial v^2}(\log\lambda) &= -4\frac{\partial}{\partial v}\left(\frac{v}{u^2+v^2+c}\right) = -4\,\frac{(u^2-v^2+c)}{(u^2+v^2+c)^2}.
\end{aligned}$$

$$\implies K = -\frac{1}{2\lambda}\Delta(\log\lambda) = -\frac{1}{2}(u^2+v^2+c)^2\left(-\frac{8c}{(u^2+v^2+c)^2}\right) = 4c.$$

This surface has constant Gaussian curvature $K = 4c$.

Remark. For $c > 0$, this correponds to the stereographic projection of a sphere of radius $1/\sqrt{c}$ minus the north pole; for $c = 0$, this corresponds to the Euclidean plane; and for $c < 0$, this corresponds to the stereographic projection of a hyperbolic plane.

Exercise 4.3.3. Verify that the surfaces

$$\mathbf{x}(u,v) = (u\cos v,\ u\sin v,\ \log u), \quad u > 0,$$

$$\bar{\mathbf{x}}(u,v) = (u\cos v,\ u\sin v,\ v),$$

have equal Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$, but that the mapping $\bar{\mathbf{x}}\circ\mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

Solution 4.3.3. First, we compute the first fundamental form of $\mathbf{x}(u,v)$ and $\bar{\mathbf{x}}(u,v)$:

$$\begin{aligned}
\mathbf{x}_u &= \left(\cos v, \sin v, \frac{1}{u}\right), \quad \mathbf{x}_v = (-u\sin v, u\cos v, 0)\,,\\
E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle = \cos^2 v + \sin^2 v + \frac{1}{u^2} = 1 + \frac{1}{u^2},\\
F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle = -u\cos v\sin v + u\sin v\cos v + 0 = 0,\\
G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\overline{\mathbf{x}}_u &= (\cos v,\ \sin v,\ 0), \quad \overline{\mathbf{x}}_v = (-u\sin v,\ u\cos v,\ 1),\\
\overline{E} &= \langle \overline{\mathbf{x}}_u,\ \overline{\mathbf{x}}_u\rangle = \cos^2 v + \sin^2 v + 0 = 1,\\
\overline{F} &= \langle \overline{\mathbf{x}}_u,\ \overline{\mathbf{x}}_v\rangle = -u\cos v\sin v + u\sin v\cos v + 0 = 0,\\
\overline{G} &= \langle \overline{\mathbf{x}}_v,\ \overline{\mathbf{x}}_v\rangle = u^2\sin^2 v + u^2\cos^2 v + 1 = u^2 + 1.
\end{aligned}$$

Notice that for orthogonal parametrizations, the Gaussian curvature only depends on the following quantities:

$$E_v = \overline{E}_v = 0, \quad G_u = \overline{G}_u = 2u, \quad EG = \left(1 + \frac{1}{u^2}\right)u^2 = u^2 + 1 = \overline{E}\,\overline{G}.$$

Since $F = \overline{F} = 0$, both parametrizations are orthogonal, so by Exercise 4.3.1 the Gaussian curvature at the points $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$ are equal. Consider the map $\Phi : S \to \overline{S}$ defined by $\Phi = \overline{\mathbf{x}}\circ\mathbf{x}^{-1}$, where S and $\overline{S}$ are the images of $\mathbf{x}$ and $\overline{\mathbf{x}}$, respectively. Since Φ satisfies $\Phi(\mathbf{x}(u,v)) = \overline{\mathbf{x}}(u,v)$, we have

$$\mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u) = \frac{\partial}{\partial u}\overline{\mathbf{x}}(u,v) = \overline{\mathbf{x}}_u, \quad \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_v) = \frac{\partial}{\partial v}\overline{\mathbf{x}}(u,v) = \overline{\mathbf{x}}_v.$$

Then, we compute the first fundamental form at $\mathbf{x}(u,v)$ under the map Φ:

$$\langle \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u),\ \mathrm{d}\Phi_{\mathbf{x}(u,v)}(\mathbf{x}_u)\rangle = \langle \overline{\mathbf{x}}_u,\ \overline{\mathbf{x}}_u\rangle = \overline{E} = 1 \neq 1 + \frac{1}{u^2} = E = \langle \mathbf{x}_u,\ \mathbf{x}_u\rangle,$$

so Φ is not an isometry.

Remark. Two regular surfaces with identical Gaussian curvature at corresponding points are not necessarily isometric.

Exercise 4.3.4. Show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

Solution 4.3.4.

Exercise 4.3.5. If the coordinate curves form a Tchebyshef net (cf. Exercises 7 and 8, Sec. 2–5), then $E = G = 1$ and $F = \cos\theta$. Show that in this case

$$K = -\frac{\theta_{uv}}{\sin\theta}.$$

Solution 4.3.5.

Remark. In principle, the Gaussian curvature is completely determined by the first fundamental form. However, in practie, it is often difficult to calculate K directly from E, F, G.

Method 1: From Theorema Egregium in Do Carmo Curves and Surfaces, we have

$$K = \frac{1}{E}\left[-\left(\Gamma_{12}^2\right)_u + \left(\Gamma_{11}^2\right)_v - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + \Gamma_{11}^1\Gamma_{12}^2\right].$$

In this case, the Christoffel symbols satisfy the following relations:

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \dfrac{E_u}{2} = 0, \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \dfrac{E_v}{2} = -\sin\theta\,\theta_u, \\ \Gamma_{12}^1 E + \Gamma_{12}^2 F = \dfrac{E_v}{2} = 0, \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \dfrac{G_u}{2} = 0, \\ \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \dfrac{G_u}{2} = -\sin\theta\,\theta_v, \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \dfrac{G_v}{2} = 0. \end{cases}$$

Then, since $|g|^2 = EG - F^2 = \sin^2\theta$, we have

$$\begin{aligned} &\Gamma_{11}^1 = \frac{1}{|g|^2} F \sin\theta\theta_u = \cot\theta\,\theta_u, \quad \Gamma_{11}^2 = -\frac{1}{|g|^2} E\sin\theta\theta_u = -\csc\theta\,\theta_u, \\ &\Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{21}^1 = \Gamma_{21}^2 = 0, \\ &\Gamma_{22}^1 = -\frac{1}{|g|^2} G\sin\theta\theta_v = -\csc\theta\,\theta_v, \quad \Gamma_{22}^2 = \frac{1}{|g|^2} F\sin\theta\theta_v = \cot\theta\,\theta_v. \end{aligned}$$

Next, $\left(\Gamma_{11}^2\right)_v = (-\csc\theta\theta_u)_v = \csc\theta\cot\theta\theta_u\theta_v - \csc\theta\theta_{uv}$. By the Theorema Egregium, we have

$$K = \csc\theta\cot\theta\theta_u\theta_v - \csc\theta\theta_{uv} - 0 - 0 + (-\csc\theta\theta_u)(\cot\theta\theta_v) + 0 = -\frac{\theta_{uv}}{\sin\theta}.$$

Method 2: From Riemannian geometry, the Theorema Egregium states that

$$R_{1212} = \langle R(\partial_u, \partial_v)\partial_u, \partial_v\rangle = -\det g\, K.$$

By antisymmetry of the curvature tensor, we have

$$K = -\frac{1}{\det g} R_{1212} = \frac{1}{\det g} R_{1221} = \frac{1}{\det g}\langle R(\partial_u, \partial_v)\partial_v, \partial_u\rangle.$$

Exercise 4.3.6. Show that there exists no surface $\mathbf{x}(u, v)$ such that

$$E = G = 1, \quad F = 0 \quad \text{and} \quad e = 1, \quad g = -1, \quad f = 0.$$

Solution 4.3.6. Suppose such a surface $\mathbf{x}(u,v)$ exists. Since $E = G = 1$ and $F = 0$, the parametrization is orthogonal. From Exercise 4.3.1, we have

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} = -\frac{1}{2}(0+0) = 0.$$

On the other hand, from the Gauss formula, we have

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(1)(-1) - 0^2}{(1)(1) - 0^2} = -1,$$

a contradiction.

Exercise 4.3.7. Does there exist a surface $\mathbf{x} = \mathbf{x}(u,v)$ with

$$E = 1, \quad F = 0, \quad G = \cos^2 u \quad \text{and} \quad e = \cos^2 u, \quad f = 0, \quad g = 1\,?$$

Solution 4.3.7.

Exercise 4.3.8. Compute the Christoffel symbols for an open set of the plane

a. In Cartesian coordinates.

b. In polar coordinates.

Use the Gauss formula to compute K in both cases.

Solution 4.3.8.

a. An open set of the plane can be parametrized in Cartesian coordinates as $\mathbf{x}(u,v) = (u, v, 0)$. Then, we have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = 1, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = 1.$$

Since $F = 0$ and $E, G \neq 0$, we have

$$\Gamma_{11}^1 = \frac{E_u}{2E} = 0, \quad \Gamma_{11}^2 = -\frac{E_v}{2G} = 0, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{E_v}{2E} = 0,$$
$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{G_u}{2G} = 0, \quad \Gamma_{22}^1 = -\frac{G_u}{2E} = 0, \quad \Gamma_{22}^2 = \frac{G_v}{2G} = 0.$$

Hence, all Christoffel symbols are zero. Next, compute

$$\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = 0,$$

so with the unit normal $N = (0,0,1)$, we have

$$e = \langle \mathbf{x}_{uu}, N\rangle = 0, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = 0.$$

Therefore, since $EG - F^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as

$$K = \frac{eg - f^2}{EG - F^2} = 0.$$

b. An open set of the plane can also be parametrized in polar coordinates, given by the parametrization $\mathbf{x}(u,v) = (u\cos v, u\sin v, 0)$. Then, we have

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = 1, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = u^2.$$

Since $F = 0$, we have the following Christoffel symbols whenever $u \neq 0$:

$$\Gamma^1_{11} = \frac{E_u}{2E} = 0, \quad \Gamma^2_{11} = -\frac{E_v}{2G} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E} = 0,$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G} = \frac{1}{u}, \quad \Gamma^1_{22} = -\frac{G_u}{2E} = -u, \quad \Gamma^2_{22} = \frac{G_v}{2G} = 0.$$

Unlike in the Cartesian coordinates, not all Christoffel symbols are zero. Next, compute

$$\mathbf{x}_{uu} = (0,0,0), \quad \mathbf{x}_{uv} = (-\sin v, \cos v, 0), \quad \mathbf{x}_{vv} = (-u\cos v, -u\sin v, 0),$$

so with the unit normal $N = (0,0,1)$, we have

$$e = \langle \mathbf{x}_{uu}, N\rangle = 0, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = 0.$$

Therefore, since $EG - F^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as

$$K = \frac{eg - f^2}{EG - F^2} = 0.$$

Exercise 4.3.9. Justify why the surfaces below are not pairwise locally isometric:

a. Sphere.

b. Cylinder.

c. Saddle $z = x^2 - y^2$.

Solution 4.3.9.

a. The sphere has constant positive Gaussian curvature. Let a sphere of radius r be centered about the origin, and let $\mathbf{x}(\theta,\phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ be a parametrization of the sphere. Then,

$$\mathbf{x}_\theta = (r\cos\theta\cos\phi,\ r\cos\theta\sin\phi,\ -r\sin\theta),$$
$$\mathbf{x}_\phi = (-r\sin\theta\sin\phi,\ r\sin\theta\cos\phi,\ 0),$$

and we have

$$E = r^2, \quad F = 0, \quad G = r^2\sin^2\theta.$$

We can compute

$$E_\phi = 0, \quad G_\theta = 2r^2\sin\theta\cos\theta, \quad EG = r^4\sin^2\theta.$$

Then,

$$\left(\frac{E_\phi}{\sqrt{EG}}\right)_\phi = 0, \quad \left(\frac{G_\theta}{\sqrt{EG}}\right)_\theta = (\cos\theta)_\theta = -\sin\theta.$$

Since $F = 0$, the parametrization is orthogonal. By Exercise 4.3.1 we have

$$K = -\frac{1}{2r^2\sin\theta}(-\sin\theta) = \frac{1}{2r^2} > 0.$$

b. The cylinder has zero Gaussian curvature. Let a cylinder of radius r be centered about the z-axis, and let $\mathbf{x}(\theta, z) = (r\cos\theta, r\sin\theta, z)$ be a parametrization of the cylinder. Then,

$$\mathbf{x}_\theta = (-r\sin\theta,\ r\cos\theta,\ 0), \quad \mathbf{x}_z = (0,\ 0,\ 1),$$

and we have

$$E = r^2, \quad F = 0, \quad G = 1.$$

We can compute

$$E_z = 0, \quad G_\theta = 0, \quad EG = r^2.$$

Then,
$$\left(\frac{E_z}{\sqrt{EG}}\right)_z = 0, \quad \left(\frac{G_\theta}{\sqrt{EG}}\right)_\theta = 0.$$
Since $F = 0$, the parametrization is orthogonal. By Exercise 4.3.1 we have
$$K = -\frac{1}{2r}(0+0) = 0.$$

c. The saddle has negative Gaussian curvature. Let the saddle be given by the parametrization $\mathbf{x}(u,v) = (u, v, u^2 - v^2)$. Then,
$$\mathbf{x}_u = (1,0,2u), \quad \mathbf{x}_v = (0,1,-2v),$$
$$\mathbf{x}_{uu} = (0,0,2), \quad \mathbf{x}_{uv} = (0,0,0), \quad \mathbf{x}_{vv} = (0,0,-2),$$
and we have $E = 1+4u^2$, $F = -4uv$, and $G = 1+4v^2$. The normal vector of the surface is given by
$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} = \frac{(-2u, 2v, 1)}{\sqrt{1+4u^2+4v^2}}.$$
Then, we have
$$e = \langle \mathbf{x}_{uu}, N\rangle = \frac{2}{\sqrt{1+4u^2+4v^2}}, \quad f = \langle \mathbf{x}_{uv}, N\rangle = 0, \quad g = \langle \mathbf{x}_{vv}, N\rangle = \frac{-2}{\sqrt{1+4u^2+4v^2}}.$$
Since $EG - F^2 = (1+4u^2)(1+4v^2) - 16u^2v^2 = 1+4u^2+4v^2 \neq 0$, the Gaussian curvature is given by the Gauss formula as
$$K = \frac{eg - f^2}{EG - F^2} = \frac{\left(\frac{2}{\sqrt{1+4u^2+4v^2}}\right)\left(\frac{-2}{\sqrt{1+4u^2+4v^2}}\right) - 0}{1+4u^2+4v^2} = \frac{-4}{(1+4u^2+4v^2)^2} < 0.$$

Suppose **a.** to **c.** are pairwise locally isometric, then by the Theorema Egregium they must have identical Gaussian curvature at corresponding points, a contradiction to our above calculation.

The following are some extra exercises from other sources.

Theorem 7 (Levi-Civita connection formula)**.**

Let g be the metric, or the first fundamental form, on a surface S. The Christoffel symbols associated to g are given by
$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right),$$
where (g^{ij}) is the inverse matrix of (g_{ij}).

Proof. Let $\{e_i\}$ be the coordinate basis induced by the parametrization $\mathbf{x}(u^1, \ldots, u^n)$. Then,
$$\partial_j e_i = \nabla_{e_j} e_i = \sum_{k=1}^n \Gamma_{ij}^k e_k \equiv \Gamma_{ij}^k e_k.$$
The metric tensor is $g_{ij} = \langle e_i, e_j\rangle$, and
$$\begin{aligned}\partial_k g_{ij} &= \partial_k \langle e_i, e_j\rangle = \langle \partial_k e_i, e_j\rangle + \langle e_i, \partial_k e_j\rangle \\ &= \langle \Gamma_{ik}^l e_l, e_j\rangle + \langle e_i, \Gamma_{jk}^m e_m\rangle = \Gamma_{ik}^l g_{lj} + \Gamma_{jk}^m g_{im}.\end{aligned}$$
By permuting the indices, we also have $\partial_j g_{ik} = \Gamma_{ij}^l g_{lk} + \Gamma_{kj}^l g_{il}$ and $\partial_i g_{jk} = \Gamma_{ji}^l g_{lk} + \Gamma_{ki}^l g_{jl}$. Recall that since $\mathbf{x}_{ij} = \mathbf{x}_{ji}$ by smoothness of $\mathbf{x}$, we have $\partial_i e_j = \partial_j e_i$, and hence $\Gamma_{ij}^k = \Gamma_{ji}^k$. Therefore, we have
$$2\Gamma_{ij}^l g_{lk} = \partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}.$$

Contract with g^{km} and use $g^{km}g_{lk} = \delta_l^m$ to obtain

$$2\Gamma_{ij}^m = g^{km}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right) \implies \Gamma_{ij}^m = \frac{1}{2}g^{km}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right).$$

$\square$

Exercise 1 (Christoffel synbols in higher dimensions)**.** Here we calculate the Christoffel symbols for various high-dimenional manifolds.

a. Hyper-paraboloid: Let $(x^1, \ldots, x^n)$ be coordinates in $\mathbb{R}^n$. Consider the immersion $\Phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by

$$\Phi\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^n, \sum_{i=1}^n (x^i)^2\right).$$

Its image is the hyper-paraboloid in $\mathbb{R}^{n+1}$. Compute the Christoffel symbols of the induced metric (from $\langle \cdot \rangle_{\mathbb{R}^{n+1}}$) on the hyper-paraboloid.

b. Conformally flat metric in $\mathbb{R}^n$: Consider the metric g on $\mathbb{R}^3$ defined by $g_{ij} = e^{2\phi(x)}\delta_{ij}$. Comput Γ_{ij}^k in terms of ϕ.

c. n-sphere: Consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$ with the parametrization

$$\mathbf{x}(u^1, \ldots, u^n) = \begin{pmatrix} \cos u^1 \\ \sin u^1 \cos u^2 \\ \sin u^1 \sin u^2 \cos u^3 \\ \vdots \\ \sin u^1 \sin u^2 \cdots \sin u^{n-1} \sin u^n \end{pmatrix},$$

where $u^1 \in [0, \pi]$, $u^2, \ldots, u^{n-1} \in [0, \pi]$, and $u^n \in [0, 2\pi)$. Compute the Christoffel symbols of the induced metric on S^n.

Solution 1.

a.

Exercise 2 (computing the Ricci tensor)**.** Let $f \in C^\infty(U)$, $f > 0$, and $g_{ij}(x^1, \ldots, x^n) = f(x^n)\delta_{ij}$. Then, calculate the Ricci tensor R_{ij} in terms of f and its derivatives.

Solution 2.

4 Chapter 4.4

Definition 2 (covariant derivative 協變導數)**.** Let w be a differentiable vector field restricted to a curve $\alpha : I \to S$. The vector by the normal projection of $\mathrm{d}w/\mathrm{d}t$ onto $T_p(S)$ is called the covariant derivative of the vector field w relative to $\alpha'(0)$.

Definition 3 (covariant derivative of vector field along a curve)**.** Let w be a differentiable vector field along a curve $\alpha : I \to S$. The expression

$$\begin{aligned}\frac{\mathrm{D}w}{\mathrm{d}t}(t) = &\left(a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv'\right)\mathbf{x}_u \\ &+ \left(b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{12}bu' + \Gamma^2_{22}bv'\right)\mathbf{x}_v\end{aligned} \tag{1}$$

is called the covariant derivative of the vector field w along the curve α.

Definition 4 (parallel vector field)**.** A vector field w along a curve $\alpha : I \to S$ is said to be parallel if $\mathrm{D}w/\mathrm{d}t = 0$ for all $t \in I$.

Definition 5 (parallel transport 平行輸運)**.** Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Let w be the (unique) parallel vector field along α such that $w(t_0) = w_0$. The vector $w(t) \in T_{\alpha(t)}(S)$ is called the parallel transport of w_0 along α at $\alpha(t)$.

Definition 6 (parametrized geodesic 參數測地線)**.** A nonconstant, parametrized curve $\gamma : I \to S$ is said to be geodesic at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t, i.e.

$$\frac{\mathrm{D}\gamma'}{\mathrm{d}t}(t) = 0. \tag{2}$$

We say γ is a parametrized geodesic if it is geodesic for all $t \in I$.

Definition 7 (geodesic 測地線)**.** A regular connected curve C in S is said to be a geodesic if, for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of p by arc length s is a parametrized geodesic. That is, $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

Definition 8. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$ is normal to $\mathrm{d}w(t)/\mathrm{d}t$, we can write

$$\frac{\mathrm{D}w}{\mathrm{d}t}(t) = \lambda(t)\left(N(t) \wedge w(t)\right), \quad \lambda(t) \equiv \left[\frac{\mathrm{D}w}{\mathrm{d}t}\right],$$

where $[\mathrm{D}w/\mathrm{d}t]$ is called the algebraic value of $\mathrm{D}w/\mathrm{d}t$ at t.

Definition 9 (geodesic curvature 測地線曲率)**.** Let $C \subset S$ be a regular curve on an oriented surface S, and let $\alpha(s)$ be its parametrization by arc length. The algebraic value of the covariant derivative $[\mathrm{D}\alpha'(s)/\mathrm{d}s] \equiv k_\mathrm{g}$ of $\alpha'(s)$ at p is called the geodesic curvature of C at $p = \alpha(s)$.

Remark. Immediately, we have $k^2 = k_n^2 + k_g^2$.

Proposition 8 (algebraic value of covariant derivative)**.**

Let $\mathbf{x}(u,v)$ be an orthonormal parametrization of a neighborhood of an oriented surface S, and $w(t)$ be a differentiable field of unit vectors along a curve $\mathbf{x}(u(t),v(t))$. Then,

$$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{\mathrm{d}v}{\mathrm{d}t} - E_v\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \frac{\mathrm{d}\phi}{\mathrm{d}t}, \tag{3}$$

where $\phi(t) = \cos^{-1}\langle \mathbf{x}_u/\sqrt{E},\, w(t)\rangle$ is the angle from $\mathbf{x}_u$ to $w(t)$ in the given orientation.

Proposition 9 (Liouville). Let $\alpha(s)$ be a parametrization by arc length of a neighborhood of p of a regular oriented curve C on an oriented surface S. Let $\mathbf{x}$ be an orthonormal parametrization of a neighborhood of p such that the angle between $\alpha'(s)$ and $\mathbf{x}_u$ is $\phi(s)$. Then,

$$k_g = (k_g)_1\cos\phi + (k_g)_2\sin\phi + \frac{\mathrm{d}\phi}{\mathrm{d}s},$$

where $(k_g)_1$ and $(k_g)_2$ are the geodesic curvatures of the coordinate curves $v =$ const. and $u =$ const., respectively.

Theorem 10 (differential equations of the geodesics).

Let $\alpha : I \to S$ be a parametrized curve on a surface S, and let $\mathbf{x}(u,v)$ be a parametrization of S in a neighborhood of $\alpha(t_0)$, $t_0 \in I$. Then, the tangent vector field $\alpha'(t)$, $t \in J$, is given by $w(t) = u'(t)\mathbf{x}_u + v'(t) + \mathbf{x}_v$. Since w is parellel along α, the functions $u(t)$, $v(t)$ satisfy

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0, \quad v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0.$$

Additional definitions for Riemannian geometry.

Definition 10 (Levi-Civita formula). The Christoffel symbols associated to the first fundamental form are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right),$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Definition 11 (connection 1-form). Given an orthonormal frame $\{e_1, e_2\}$ on a surface S, the connection 1-form ω is defined by

$$\omega(X) = \langle \nabla_X e_1, e_2\rangle,$$

for any vector field X on S.

Definition 12 (Levi-Civita connection). The Christoffel symbols defined by the Levi-Civita formula determine a unique connection ∇ on the tangent bundle of a surface S, called the Levi-Civita connection. This is given by the formula

$$\nabla_{e_i} e_j = \Gamma^k_{ij}\left(e_1, \ldots, e_n\right) e_k.$$

Definition 13 (Riemannian tensor). Given vector fields X, Y, Z on S, the Riemmannian curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

In coordinates, we have

$$R^l{}_{ijk} = \langle R(\partial_i,\partial_j)\partial_k, \partial_l\rangle = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

Definition 14 (Ricci tensor & scalar curvature)**.** The Ricci tensor R_{ij} is defined by contracting the Riemannian curvature tensor as

$$R_{ij} = R^k{}_{ikj}.$$

Then, the scalar curvature R is defined as the trace of the Ricci tensor, i.e. $R = g^{ij}R_{ji}$.

Exercise 4.4.1.

a. Show that if a curve $C \subset S$ is both a line of curvature and a geodesic, then C is a plane curve.

b. Show that if a (nonrectilinear) geodesic is a plane curve, then it is a line of curvature.

c. Give an example of a line of curvature which is a plane curve and not a geodesic.

Solution 4.4.1.

a. By Proposition 3.2.3, the theorem of Olinde Rodrigues states that a regular curve is a line of curvature if and only if the normal vector N along C satisfies $N'(t) = \lambda(t)\alpha'(t)$ for some function λ. Since $\alpha(t)$ is a geodesic, we have $\mathrm{D}\alpha'/\mathrm{dt} = \mathrm{D}T/\mathrm{dt} = 0$, and thus $T'(t) = \mu(t)N(t)$ for some differentiable function μ. Therefore, the binormal vector $B(t) = T(t) \wedge N(t)$ satisfies

$$B'(t) = T'(t) \wedge N(t) + T(t) \wedge N'(t) = \mu(t)N(t) \wedge N(t) + T(t) \wedge \lambda(t)\alpha'(t) = 0.$$

Then, $\frac{\mathrm{d}}{\mathrm{dt}}\langle \alpha(t), B(t)\rangle = \langle T(t), B(t)\rangle + \langle \alpha(t), B'(t)\rangle = 0$, and C is a plane curve.

b. Suppose C is a geodesic and a plane curve. Then, the normal vector N along C is constant, so $N'(t) = 0$. Since $\alpha(t)$ is a geodesic, we have $\langle N'(t),\, \alpha'(t)\rangle = 0$, *

c. Let C be the curve of constant latitude on a sphere S with latitude $0 < \phi < \pi/2$. Then, C is a line of curvature since the normal vector along C is constant. Also, C is a plane curve since it lies in a plane parallel to the equatorial plane. However, C is not a geodesic since the geodesics on a sphere are exactly the great circles.

Exercise 4.4.2. Prove that a curve $C \subset S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line.

Solution 4.4.2. Suppose C is both an asymptotic curve and a geodesic, and C is the trace of the parametrization $\alpha : I \to \mathbb{R}^3$. Then $k_n = k_g = 0$. Thus, $k^2 = k_n^2 + k_g^2 = 0$ implies $k = 0$, and so $\alpha'' = kn = 0$. Integrating twice, we have $\alpha(t) = at+b$, a straight line. Conversely, if C is a straight line, then $kn = \alpha'' = 0$. Taking the norm on both sides shows $k = 0$, and hence $k_g = k_n = 0$.

Exercise 4.4.3. Show, without using Prop. 5, that the straight lines are the only geodesics of a plane.

Solution 4.4.3. For a plane, the unit normal N is constant, and thus $\mathrm{d}N = 0$. Therefore, the second fundamental form $\mathrm{II}(v, w) = -\langle \mathrm{d}N_p(v),\, w\rangle N = 0$ for all $v, w \in T_p(S)$, and k_n is identically zero. For a geodesic, we have $k_g = 0$, so $k = 0$, and $\alpha'' = 0$. Integrating twice, we have $\alpha(t) = at+b$, a straight line. Conversely, a straight line has $\alpha'' = 0$, so $k = 0$, and hence $k_g = 0$.

Exercise 4.4.4. Let v and w be vector fields along a curve $\alpha : I \to S$. Prove that

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \left\langle \frac{Dv}{dt},\, w(t)\right\rangle + \left\langle v(t),\, \frac{Dw}{dt}\right\rangle.$$

Solution 4.4.4. The covariant derivative is the normal projection of the ordinary derivative onto the tangent space. Thus, we have

$$\frac{\mathrm{D}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t} - \left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, N \right\rangle N, \quad \frac{\mathrm{D}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}t} - \left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, N \right\rangle N.$$

$$\begin{aligned}
\implies \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}, w \right\rangle + \left\langle v, \frac{\mathrm{D}w}{\mathrm{d}t} \right\rangle &= \left\langle \frac{\mathrm{d}v}{\mathrm{d}t} - \left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, N \right\rangle N, w \right\rangle + \left\langle v, \frac{\mathrm{d}w}{\mathrm{d}t} - \left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, N \right\rangle N \right\rangle \\
&= \left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, v \right\rangle + \left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, w \right\rangle - \left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, N \right\rangle \langle N, w \rangle - \left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, N \right\rangle \langle v, N \rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle v, w \rangle - \frac{\mathrm{d}}{\mathrm{d}t}\langle v, N \rangle \langle N, w \rangle
\end{aligned}$$

Exercise 4.4.5. Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2, \qquad y = 0,$$

about the z axis ($a > r > 0$). The parallels generated by the points $(a+r, 0)$, $(a-r, 0)$, (a, r) are called the maximum parallel, the minimum parallel, and the upper parallel, respectively. Check which of these parallels is

a. A geodesic.

b. An asymptotic curve.

c. A line of curvature.

Solution 4.4.5. Take the standard parametrization of the torus of rotation:

$$\mathbf{x}(u, v) = ((a + r\cos v)\cos u,\ (a + r\cos v)\sin u,\ r\sin v), \quad u, v \in [0, 2\pi).$$

Then, we have

$$\begin{aligned}
\mathbf{x}_u &= (-(a + r\cos v)\sin u,\ (a + r\cos v)\cos u,\ 0), \\
\mathbf{x}_v &= (-r\sin v\cos u,\ -r\sin v\sin u,\ r\cos v), \\
\mathbf{x}_{uu} &= (-(a + r\cos v)\cos u,\ -(a + r\cos v)\sin u,\ 0), \\
\mathbf{x}_{uv} &= (r\sin v\sin u,\ -r\sin v\cos u,\ 0), \\
\mathbf{x}_{vv} &= (-r\cos v\cos u,\ -r\cos v\sin u,\ -r\sin v).
\end{aligned}$$

The first fundamental form is given by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (a + r\cos v)^2, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = r^2,$$

the unit normal is $N = (\cos v \cos u,\ \cos v \sin u,\ \sin v)$, and the second fundamental form is given by

$$e = \langle N, \mathbf{x}_{uu} \rangle = -(a + r\cos v)\cos v, \quad f = \langle N, \mathbf{x}_{uv} \rangle = 0, \quad g = \langle N, \mathbf{x}_{vv} \rangle = -r.$$

Now, we proceed to calculate the geodesic curvature k_g for each parallel with $v = \phi_0$. The maximum, minimum, and upper parallels correspond to $\phi_0 = 0$, π, and $\pi/2$, respectively. The unit tangent along the parallel is $T = \mathbf{x}_u/\sqrt{E}$, and the normal curvature is given by

$$k_n = \frac{\mathrm{II}(T, T)}{\mathrm{I}(T)} = \frac{e}{E} = -\frac{(a + r\cos\phi_0)\cos\phi_0}{(a + r\cos\phi_0)^2} = -\frac{\cos\phi_0}{a + r\cos\phi_0}.$$

We have

$$\frac{\mathrm{D}T}{\mathrm{d}s} = \Gamma^v_{uu}(T^u)^2 e_v = \Gamma^v_{uu}\frac{1}{E}\mathbf{x}_v =$$

Exercise *4.4.6. Compute the geodesic curvature of the upper parallel of the torus of Exercise 5.

Solution 4.4.6.

Exercise *4.4.8. Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.

Solution 4.4.8. Let C be a geodesic of S, and $\alpha(t)$ be its parametrization. Since C is a plane curve, we have $B' = T' \wedge N + T \wedge N' = 0$. Since C is a geodesic, we have $k_g = 0$, and thus $\alpha'' = T' = k_n N$. Hence, $T \wedge N' = 0$, and so $N' = \lambda T$ for some function λ. By Proposition 3.2.3 (Olinde Rodrigues), every point of C is an umbilical point. By Proposition 4.4.5, for any $p \in S$ and $w \in T_p(S)$, there is a unique parametrized geodesic $\gamma : I \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = w$, and hence every point of S is umbilical. Since S is connected and all its points are umbilical points, by Proposition 3.2.4 (a surface S is contained in a plane or a sphere if S is connected and all its points are umbilical points), S is contained in a plane or a sphere.

Exercise *4.4.9. Consider two meridians of a sphere C_1 and C_2 which make an angle φ at the point p_1. Take the parallel transport of the tangent vector w_0 of C_1, along C_1 and C_2, from the initial point p_1 to the point p_2 where the two meridians meet again, obtaining, respectively, w_1 and w_2. Compute the angle from w_1 to w_2.

Solution 4.4.9. Let C_1 and C_2 be the two meridians of the sphere intersecting at p_1 and $=_2$, parametrized by α_1 and α_2 respectively. Without loss of generality, let $p_1 = (0, 0, 1)$ and $p_2 = (0, 0, -1)$. Choose coordinates such that

$$\alpha_1(s) = (\sin s,\, 0,\, \cos s)\,, \quad \alpha_2(s) = (\cos\phi \sin s,\, \sin\phi \sin s,\, \cos s)\,,$$

for $0 \leq s < \pi$. We have $w_0 = \alpha_1'(0) = (1, 0, 0)$ and the transport along C_1 is $w_1(\pi) = \alpha_1'(\pi) = (-1, 0, 0)$.

Exercise *4.4.10. Show that the geodesic curvature of an oriented curve $C \subset S$ at a point $p \in C$ is equal to the curvature of the plane curve obtained by projecting C onto the tangent plane $T_p(S)$ along the normal to the surface at p.

Exercise *4.4.12. We say that a set of regular curves on a surface S is a <u>differentiable family of curves</u> on S if the tangent lines to the curves of the set make up a differentiable field of directions (see Sec. 3–4). Assume that a surface S admits two differentiable orthogonal families of geodesics. Prove that the Gaussian curvature of S is zero.

Exercise *4.4.13. Let V be a connected neighborhood of a point p of a surface S, and assume that the parallel transport between any two points of V does not depend on the curve joining these two points. Prove that the Gaussian curvature of V is zero.

Exercise 4.4.14. Let S be an oriented regular surface and let $\alpha : I \to S$ be a curve parametrized by arc length. At the point $p = \alpha(s)$ consider the three unit vectors (the Darboux trihedron)

$$T(s) = \alpha'(s), \qquad N(s) = \text{the normal vector to } S \text{ at } p, \qquad V(s) = N(s) \wedge T(s).$$

Show that

$$\frac{dT}{ds} = 0 + aV + bN,$$
$$\frac{dV}{ds} = -aT + 0 + cN,$$
$$\frac{dN}{ds} = -bT - cV + 0,$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$, $s \in I$. The above formulas are the analogues of Frenet's formulas for the trihedron T, V, N. To establish the geometrical meaning of the coefficients, prove that

- **a.** $c = -\langle dN/ds, V\rangle$; conclude from this that $\alpha(I) \subset S$ is a line of curvature if and only if $c \equiv 0$ ($-c$ is called the geodesic torsion of α; cf. Exercise 19, Sec. 3–2).
- **b.** b is the normal curvature of $\alpha(I) \subset S$ at p.
- **c.** a is the geodesic curvature of $\alpha(I) \subset S$ at p.

Solution 4.4.14. First, we show the Darboux trihedron analogue for Frenes formulas. Since T, V, N are orthonormal, we have $\langle T, T\rangle = \langle V, V\rangle = \langle N, N\rangle = 1$ and $\langle T, V\rangle = \langle V, N\rangle = \langle N, T\rangle = 0$. Differentiating these equations with respect to s, we have

$$\left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, T\right\rangle = \left\langle \frac{\mathrm{d}V}{\mathrm{d}s}, V\right\rangle = \left\langle \frac{\mathrm{d}N}{\mathrm{d}s}, N\right\rangle = 0,$$

and

$$\left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, V\right\rangle + \left\langle T, \frac{\mathrm{d}V}{\mathrm{d}s}\right\rangle = 0, \quad \left\langle \frac{\mathrm{d}V}{\mathrm{d}s}, N\right\rangle + \left\langle V, \frac{\mathrm{d}N}{\mathrm{d}s}\right\rangle = 0, \quad \left\langle \frac{\mathrm{d}N}{\mathrm{d}s}, T\right\rangle + \left\langle N, \frac{\mathrm{d}T}{\mathrm{d}s}\right\rangle = 0.$$

Hence, let $a(s) = \langle \mathrm{d}T/\mathrm{d}s, V\rangle$, $b(s) = \langle \mathrm{d}T/\mathrm{d}s, N\rangle$, and $c(s) = -\langle \mathrm{d}N/\mathrm{d}s, V\rangle$, we have

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}s} &= \left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, V\right\rangle V + \left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, N\right\rangle N = 0 + aV + bN, \\
\frac{\mathrm{d}V}{\mathrm{d}s} &= \left\langle \frac{\mathrm{d}V}{\mathrm{d}s}, T\right\rangle T + \left\langle \frac{\mathrm{d}V}{\mathrm{d}s}, N\right\rangle N \\
&= -\left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, V\right\rangle T + 0 - \left\langle \frac{\mathrm{d}N}{\mathrm{d}s}, V\right\rangle N = -aT + 0 + cN, \\
\frac{\mathrm{d}N}{\mathrm{d}s} &= \left\langle \frac{\mathrm{d}N}{\mathrm{d}s}, T\right\rangle T + \left\langle \frac{\mathrm{d}N}{\mathrm{d}s}, V\right\rangle V = -bT - cV + 0.
\end{aligned}$$

- **a.** $c(s)$ is as we defined above. By Proposition 3.2.3 (Olinde Rodrigues), $\alpha(I) \subset S$ is a line of curvature if and only if $N'(s) = \lambda(s)T(s)$ for some function λ, if and only if $c(s) = -\langle N'(s), V(s)\rangle = 0$ for all $s \in I$.
- **b.** Since $k_n = k\cos\theta$, where $\cos\theta = \langle n, N\rangle$, we have $k_n = \langle \alpha'', N\rangle$. By the first formula, $\alpha'' = \mathrm{d}T/\mathrm{d}s = aV + bN$, so $k_n = \langle aV + bN, N\rangle = b$.
- **c.** The geodesic curvature k_g is the algebraic value of the covariant derivative of $\alpha'(t)$. For a unit vector field $w(t)$ along $\alpha(t)$, we have

  $$\left[\frac{\mathrm{D}w}{\mathrm{d}t}\right] = \left\langle \frac{\mathrm{d}w}{\mathrm{d}t}, N \wedge w\right\rangle.$$

  Let $w(t) = \alpha'(t) = T(t)$, we have

  $$k_g(t) = \left[\frac{\mathrm{D}\alpha'}{\mathrm{d}t}\right] = \left\langle \frac{\mathrm{d}T}{\mathrm{d}s}, N \wedge T\right\rangle = \langle aV + bN, V\rangle = a(t).$$

Exercise 4.4.15. Let p_0 be a pole of a unit sphere S^2 and q, r be two points on the corresponding equator in such a way that the meridians p_0q and p_0r make an angle θ at p_0. Consider a unit vector v tangent to the meridian p_0q at p_0, and take the parallel transport of v along the closed curve made up by the meridian p_0q, the parallel qr, and the meridian rp_0 (Fig. 4–21).

a. Determine the angle of the final position of v with v.

b. Do the same thing when the points r, q instead of being on the equator are taken on a parallel of colatitude φ (cf. Example 1).

Solution 4.4.15.

Exercise *4.4.16. Let p be a point of an oriented surface S and assume that there is a neighborhood of p in S all points of which are parabolic. Prove that the (unique) asymptotic curve through p is an open segment of a straight line. Give an example to show that the condition of having a neighborhood of parabolic points is essential.

Solution 4.4.16.

Exercise *4.4.18. Consider a geodesic which starts at a point p in the upper part $(z > 0)$ of a hyperboloid of revolution $x^2 + y^2 - z^2 = 1$ and makes an angle θ with the parallel passing through p in such a way that $\cos\theta = 1/r$, where r is the distance from p to the z axis. Show that by following the geodesic in the direction of decreasing parallels, it approaches asymptotically the parallel $x^2 + y^2 = 1,\ z = 0$ (Fig. 4–22).

Exercise *4.4.19. Show that when the differential equations (4) of the geodesics are referred to the arc length then the second equation of (4) is, except for the coordinate curves, a consequence of the first equation of (4).

5 Chapter 4.5

Theorem 11 (turning tangents)**.**

$$\sum_{i=0}^{k}(\phi(t_{i+1})-\phi(t_i))+\sum_{i=0}^{k}\theta_i=\pm 2\pi,$$

where the sign plus or minus depends on the orientation of the curve.

Definition 15. Let S be an oriented surface. A region $R \subseteq S$ is called a simple region if R is homeomorphic to a disk and the boundary ∂R of R is the trace of a simple, closed, piecewise regular, parametrized curve $\alpha : I \to S$. Further, let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ be a parametrization and let R be bounded. Then, if f is a differentiable function on S, the integral of f over R is given by

$$\iint_R \mathrm{d}\sigma\, f = \iint_{\mathbf{x}^{-1}(R)} \mathrm{d}u\, \mathrm{d}v\, f(\mathbf{x}(u,v))\sqrt{EG-F^2},$$

and this definition is independent of the parametrization $\mathbf{x}$ chosen.

Theorem 12 (local Gauss-Bonnet Theorem)**.** Let $\mathbf{x} : U \to S$ be an isothermal parametrization of an oriented surface S, where U is homeomorphic to an open disk and $\mathbf{x}$ is compatible with the orientation of S. Let $R \subseteq \mathbf{x}(U)$ be a simple region and $\alpha : I \to S$ be such that $\alpha(I) = \partial R$. Assume α is positively oriented, parametrized by arc length, and that $\alpha(s_0), \ldots, \alpha(s_k)$ and $\theta_0, \ldots, \theta_k$ are the vertices and exterior angles of α, respectively. Then,

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \mathrm{d}s\, k_g + \iint_R \mathrm{d}\sigma\, K + \sum_{i=0}^{k}\theta_i = 2\pi,$$

where k_g is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S.

Theorem 13 (global Gauss-Bonnet Theorem)**.** Let $R \subseteq S$ be a regular region of an oriented surface S and let $C_0, \ldots, C_n$ be the closed, simple, piecewise regular curves which make up ∂R. Suppose each C_i is positively oriented and let $\{\theta_1, \ldots, \theta_p\}$ be the set of the curves $C_1, \ldots, C_n$. Then,

$$\sum_{i=1}^{n}\int_{C_i} \mathrm{d}s\, k_g + \iint_R \mathrm{d}\sigma\, K + \sum_{j=1}^{p}\theta_j = 2\pi\chi(R),$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals over each regular arc of C_i.

Corollary 14. If R is a simple region, then

$$\sum_{i=1}^{n}\int_{C_i} \mathrm{d}s\, k_g + \iint_R \mathrm{d}\sigma\, K + \sum_{j=1}^{p}\theta_j = 2\pi.$$

Corollary 15. If S is an orientable compact surface, then

$$\iint_S \mathrm{d}\sigma\, K + \sum_{j=1}^{p}\theta_j = 2\pi\chi(S).$$

Corollary 16 (interior angles of a geodesic triangle)**.** Let T be a geodesic triangle in an oriented surface S. Assume the Gaussian curvature K does not change sign in T, and let ϕ_i denote the interior angles of T. Then,

$$\sum_{i=1}^{3} \phi_i = \pi + \iint_T \mathrm{d}\sigma\, K.$$

Definition 16 (index of a vector field)**.** Let v be a differentiable vector field on a surface S. A point $p \in S$ is called a singular point of v if $v(p) = 0$. A singular point p is said to be isolated if there exists a neighborhood $U \subset S$ of p such that p is the only singular point of v in U. Let $\mathbf{x} : U \to S$ be an orthogonal parametrization of S at $p = \mathbf{x}(0,0)$ compatible with S, and let $\alpha : [0,l] \to S$ be a simple, closed, positively oriented, piecewise regular curve such that $\alpha([0,l]) \subseteq \mathbf{x}(U)$ is the boundary of a simple region R containing p and no other singular points of v.

Exercise 4.5.1. Let $S \subset \mathbb{R}^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

Solution 4.5.1. By corollary of the global Gauss–Bonnet theorem for orientable compact surfaces, we have

$$\iint_S K\, d\sigma = 2\pi\chi(S) \le 0,$$

since compact surfaces in $\mathbb{R}^3$ have Euler–Poincaré characteristic less than or equal to zero unless they are homeomorphic to a sphere. By a previous result, every compact surface in $\mathbb{R}^3$ has an elliptics point, so $K(p) > 0$ for some p. Suppose there are no points with $K < 0$, then by continuity of K there is an open neighborhood $U \subset S$ of p such that $K(q) > 0$ for all $q \in U$. Thus,

$$\iint_S K\, d\sigma = \iint_U K\, d\sigma + \iint_{S\setminus U} K\, d\sigma > 0,$$

a contradiction. Finally, since S is connected and K is a continuous mapping, there exists $r \in S$ such that $K(r) = 0$ by the Intermediate Value Theorem.

Exercise 4.5.2. Let T be a torus of revolution. Describe the image of the Gauss map of T and show, without using the Gauss-Bonnet theorem, that

$$\iint_T K\, d\sigma = 0.$$

Compute the Euler–Poincaré characteristic of T and check the above result with the Gauss–Bonnet theorem.

Solution 4.5.2. The torus of revolution T can be parametrized by

$$\mathbf{x}(u,v) = \left((a + r\cos v)\cos u,\, (a + r\cos v)\sin u,\, r\sin v\right),$$

where $a > r > 0$, $u \in [0, 2\pi)$, and $v \in [0, 2\pi)$. Then $\mathbf{x}_u = (-(1 + r\cos v)\sin u,\, (1 + r\cos v)\cos u, 0)$, $\mathbf{x}_v = (-r\sin v\cos u,\, -r\sin v\sin u, r\cos v)$. The Gauss map $N : T \to S^2$ is given by

$$N(u,v) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = (\cos v\cos u,\, \cos v\sin u,\, \sin v).$$

The image of N is the entire unit sphere S^2, since for every $(x,y,z) \in S^2$, we can find $(u,v) \in [0,2\pi) \times [0,2\pi)$ such that $N(u,v) = (x,y,z)$. The Gaussian curvature of T is given by

$$K(u,v) = \frac{\langle N_u \wedge N_v, N\rangle}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{\cos v}{r(a + r\cos v)}.$$

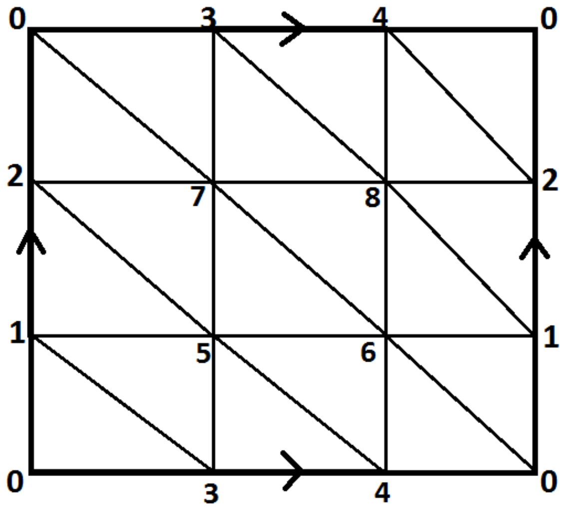

Figure 1: Triangulation of the torus.

Then, we can directly compute

$$\begin{aligned}\iint_T \mathrm{d}\sigma\, K &= \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}u\,\mathrm{d}v\, K(u,v)\sqrt{EG-F^2} \\ &= \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}u\,\mathrm{d}v\, \frac{\cos v}{r(a+r\cos v)} r(a+r\cos v) \\ &= \int_0^{2\pi}\mathrm{d}u \int_0^{2\pi} \mathrm{d}v\, \cos v = 0.\end{aligned}$$

To compute the Euler-Poincaé characteristic of T, note that the torus is isomorphic to the quotient of a square by identifying the opposite sides and identifying the vertices to a single point. Consider the triangulation of T shown in Figure 1, which has $V = 9$, $E = 27$, and $F = 18$. In fact, the minimal triangulation only has $V = 7$, $E = 21$, and $F = 14$. Thus, $\chi(T) = E - V + F = 0$.

By the global Gauss-Bonnet Theorem, we have

$$\iint_T \mathrm{d}\sigma\, K = 0 = 2\pi\chi(T) \implies \chi(T) = 0.$$

Remark. Calculating Gaussian curvature for a surface of revolution:

Exercise 4.5.3. Let $S \subset \mathbb{R}^3$ be a regular compact surface with $K > 0$. Let $\Gamma \subset S$ be a simple closed geodesic in S, and let A and B be the regions of S which have Γ as a common boundary. Let $N : S \to S^2$ be the Gauss map of S. Prove that $N(A)$ and $N(B)$ have the same area.

Solution 4.5.3.

Exercise 4.5.4. Compute the Euler–Poincaré characteristic of

a. an ellipsoid;

***b.** the surface

$$S = \{(x, y, z) \in \mathbb{R}^3\,;\ x^2 + y^{10} + z^6 = 1\}.$$

Solution 4.5.4.

a. Let E be the ellipsoid given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a, b, c > 0$. Since the linear map $L : S^2 \to E$ given by $L(x, y, z) = (ax, by, cz)$ is a diffeomorphism, $E \approx S^2$, and $\chi(E) = 2$.

b. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x, y, z) = x^2 + y^{10} + z^6$. 1 is said to be a regular point if $\mathrm{d}F_p \neq 0$, or, equivalently, $\nabla F \neq 0$. Since $\nabla F = (2x, 10y^9, 6z^5) = 0$ only at $(0, 0, 0)$, which is not in S, we have that 1 is a regular value of F. By the Regular Value Theorem, $S = F^{-1}(1)$ is a regular surface. Since $\{1\} \subseteq \mathbb{R}$ is closed and F is continuous, $S = F^{-1}(\{1\})$ is closed. Furthermore, we have $|x|, |y|, |z| \leq 1$, so S is bounded. By the Heine-Borel Theorem, S is compact. Moreover, S is orientable with $N = \nabla F/|\nabla F|$. For fixed $u = (u_1, u_2, u_3) \in S^2$, let $\phi_u : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\phi_u(r) \equiv F(ru)$ for $r > 0$. Since ϕ_u is continuous, $\phi_u'(r) = 2ru_1^2 + 10r^9u_2^{10} + 6r^5u_3^6 > 0$, and $\phi_(0) = 0$, $\phi_u(\infty) = \infty$, by the Intermediate Value Theorem there exists a unique $r_u > 0$ such that $\phi_u(r_u) = 1$.

Claim. The map $\psi : S^2 \to S$ given by $\psi(u) = r_u u$ is a continuous bijection.

Proof. Deifne $G : (0, \infty) \times S^2 \to \mathbb{R}$ by $G(r, u) = F(ru) - 1$. Then $G(r_u, u) = 0$ and

$$\frac{\partial G}{\partial u}(r_u, u) = \langle \nabla F(r_u u), u\rangle = 2r_u u_1^2 + 10r_u^9 u_2^{10} + 6r_u^5 u_3^6 > 0.$$

Hence, by the Implicit Function Theorem, r_u depends smoothly on u, and thus $\psi = u_r u$ is continuous. For $p \in S$, let $u = p/\|p\|$, then $F(\|p\|u) = F(p) = 1$, and by the uniqueness of r_u, $r_u = \|p\|$. Then $\psi(u) = r_u u = p$, and ψ is surjective. Let $p \in S$ satisfy $\psi(p) = r_p p = 0$. Since $r_p > 0$, it must be that $p = 0$, hence ψ is injective, and hence a bijection. □

Theorem 17. A continuous bijection between a compact space and a Hausdorff space is a homeomorphism.

Since ψ is a continuous bijection between a compact space S^2 and a Hausdorff space S, by the theorem above ψ is a homeomorphism, and thus $S \approx S^2$. Therefore, $\chi(S) = 2$.

Exercise 4.5.5. Let C be a parallel of colatitude φ on an oriented unit sphere S^2, and let w_0 be a unit vector tangent to C at a point $p \in C$ (cf. Example 1, Sec. 4–4). Take the parallel transport of w_0 along C and show that its position, after a complete turn, makes an angle

$$\Delta\varphi = 2\pi(1 - \cos\varphi)$$

with the initial position w_0. Check that

$$\lim_{R\to p} \frac{\Delta\varphi}{A} = 1 = \text{curvature of } S^2,$$

where A is the area of the region R of S^2 bounded by C.

Solution 4.5.5. Let S^2 be the unit sphere parametrized by

$$\mathbf{x}(u, v) = (\sin v \cos u,\ \sin v \sin u,\ \cos v),$$

where $u \in [0, 2\pi)$ and $v \in [0, \pi]$. Then, the parallel of colatitude φ is given by $C : \alpha(t) = (\sin\varphi \cos t,\ \sin\varphi \sin t,\ \cos\varphi)$, where $t \in [0, 2\pi]$. The tangent vector to C at $p = \alpha(0)$ is given by

$$w_0 = \alpha'(0) = (0, \sin\varphi, 0).$$

We have $\alpha'(t) = (-\sin\varphi \sin t,\ \sin\varphi \cos t,\ 0)$, $\alpha''(t) = (-\sin\varphi \cos t,\ -\sin\varphi \sin t,\ 0)$, and $\langle \alpha \wedge \alpha', \alpha''\rangle = \sin^2\varphi \cos\varphi$. The geodesic curvature of C is given by

$$k_g(t) = \frac{\langle N(\alpha(t)) \wedge \alpha'(t),\ \alpha''(t)\rangle}{\|\alpha'(t)\|^3} = \frac{\langle \alpha(t) \wedge \alpha'(t),\ \alpha''(t)\rangle}{\|\alpha'(t)\|^3} = \cot\varphi,$$

where the Gauss map N of S^2 satisfies $N(\alpha(t)) = \alpha(t)$. Now, we can compute the parallel transport of w_0 along C. Let $e_1 = \mathbf{x}_u/\|\mathbf{x}_u\| = \mathbf{x}_u/\sin v$, $e_2 = v$ be the orthonormal tangent frame. Along the paralle $v = \varphi$, we can write

$$\alpha(t) = \mathbf{x}_u(t, \varphi) = \sin\varphi\, e_1(t).$$

Let the paralle transport of w_0 along C be given by $w(t) = a(t)e_1(t) + b(t)e_2(t)$, where $a(0) = 1$, $b(0) = 0$. Then,

$$\frac{\mathrm{D}w}{\mathrm{d}t} \implies *$$

$$\Delta\varphi = \int_0^{2\pi} k_g(t)\, dt = \int_0^{2\pi} \sin\varphi\, dt = 2\pi(1 - \cos\varphi).$$

The area of the region R bounded by C is given by

$$A = \iint_R K\, d\sigma = 2\pi(1 - \cos\varphi),$$

since the Gaussian curvature of the unit sphere is identically equal to one. Thus,

$$\lim_{R\to p} \frac{\Delta\varphi}{A} = \lim_{\varphi\to 0} \frac{2\pi(1 - \cos\varphi)}{2\pi(1 - \cos\varphi)} = 1,$$

which is the curvature of S^2.

Exercise *4.5.6. Show that $(0, 0)$ is an isolated singular point and compute the index at $(0, 0)$ of the following vector fields in the plane:

***a.** $v = (x, y)$;

b. $v = (-x, y)$;

c. $v = (x, -y)$;

***d.** $v = (x^2 - y^2, -2xy)$;

e. $v = (x^3 - 3xy^2, y^3 - 3x^2y)$.

Solution 4.5.6.

a. Since $v(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$, $(0, 0)$ is an isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (\cos t, \sin t)$, and the angle between $v(\alpha(t))$ and the positive x-axis is just t. Thus,

$$\operatorname{ind}(v; (0, 0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t = 1.$$

b. Since $v(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$, $(0, 0)$ is an isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (-\cos t, \sin t)$, and the angle between $v(\alpha(t))$ and the positive x-axis is $\pi - t$. Thus,

$$\operatorname{ind}(v; (0, 0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t\, (-1) = -1.$$

c. Since $v(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$, $(0, 0)$ is an isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (\cos t, -\sin t)$, and the angle between $v(\alpha(t))$ and the positive x-axis is $-t$. Thus,

$$\operatorname{ind}(v; (0, 0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t\, (-1) = -1.$$

d. Suppose $v(x,y) = (0,0)$, then $-2xy = 0$ and one of x, y must be zero. If $x = 0$, then $x^2 - y^2 = -y^2 = 0$ implies $y = 0$, and similarly for $y = 0$. Thus, $(0,0)$ is an isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (\cos^2 t - \sin^2 t, -2\cos t \sin t) = (\cos 2t, -\sin 2t)$, and the angle between $v(\alpha(t))$ and the positive x-axis is $-2t$. Thus, the index of v at $(0,0)$ is

$$\mathrm{ind}(v; (0,0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \, (-2) = -2.$$

e. Suppose $v(x,y) = (0,0)$, then $y^3 - 3x^2 y = y(y^2 - 3x^2) = 0$ and either $y = 0$ or $y^2 = 3x^2$. If $y = 0$, then $x^3 - 3xy^2 = x^3 = 0$ implies $x = 0$. If $y^2 = 3x^2$, then substituting into the first equation gives $x^3 - 3x(3x^2) = x^3 - 9x^3 = -8x^3 = 0$, so $x = 0$ and thus $y = 0$. Therefore, $(0,0)$ is an isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (\cos^3 t - 3\cos t \sin^2 t, \sin^3 t - 3\cos^2 t \sin t) = (\cos 3t, \sin 3t)$, and the angle between $v(\alpha(t))$ and the positive x-axis is $3t$. Thus,

$$\mathrm{ind}(v; (0,0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \, 3 = 3.$$

Exercise 4.5.7. Can it happen that the index of a singular point is zero? If so, give an example.

Solution 4.5.7.

Remark. Intuitively, the index of a singular point defines the idea of how many times the vector field "turns around" when we go around a small loop enclosing the singular point. If the vector field does not turn at all, then the index is zero.

Yes. Let $v(x,y) = (x^2 + y^2, 0)$, then $v = 0$ if and only if $x = y = 0$, so $(0,0)$ is a isolated singular point. Consider the circle $C : \alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then, $v(\alpha(t)) = (1, 0)$, and the angle between $v(\alpha(t))$ and the positive x-axis is 0 for all t. Thus,

$$\mathrm{ind}(v; (0,0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \, 0 = 0.$$

Exercise 4.5.8. Prove that an orientable compact surface $S \subset \mathbb{R}^3$ has a differentiable vector field without singular points if and only if S is homeomorphic to a torus.

Solution 4.5.8. By the Poincareé-Hopf Theorem, we have

$$\sum_{i=1}^{n} \mathrm{ind}(v; p_i) = \chi(S),$$

where $p_1, \ldots, p_n$ are the isolated singular points of v. If S has a differentiable vector field without singular points, then the left-hand side is zero, so $\chi(S) = 0$. By the classification theorem of compact surfaces, the only orientable compact surface with Euler-Poincaré characteristic zero is, up to homeomorphism, the torus. Conversely, let S be homeomorphic to a torus. Let $\mathbf{x}(u,v) = ((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin v)$ be (the parametrization of) the standard torus of revolution T, then the coordinate vector field

$$\mathbf{x}_u = (-r\sin u \cos v,\ -r\sin u \sin v,\ r\cos u)\,, \quad \|\mathbf{x}_u\| = r > 0$$

never vanishes. Hence, T has a differentiable vector field without singular points.

Exercise 4.5.9. Let C be a regular closed simple curve on a sphere S^2. Let v be a differentiable vector field on S^2 with isolated singularities such that the trajectories of v are never tangent to C. Prove that each of the two regions determined by C contains at least one singular point of v.

Solution 4.5.9. By the Jordan Curve Theorem, $S \setminus C$ is divided into two simple connected regions R_1 and R_2, $\partial R_1 = \partial R_2 = C$, and $\overline{R}_1, \overline{R}_2 \approx D$ the unit disk. Hence, $\chi(R_i) = 1$, $i = 1, 2$. Suppose no trajectory of v is tangent to C, so $v(p) \notin T_p(C)$ for all p. At points p along C, choose the normal $N_i \in T_p(S^2)$ pointing outwards from R_i. Let $\phi(p) = \langle v(p), N_i(p) \rangle$, then $\phi(p) \neq 0$ for all $p \in C$. Since C is connected and ϕ is continuous, $\phi(p)$ has constant sign on C. Without loss of generality, assume $\phi(p) > 0$ for all $p \in C$. Take v or $-v$ to make it point everywhere outwards on C, then we can apply the Poincaré-Hopf Theorem to R_i:

$$\sum_{j=1}^{n_i} \mathrm{ind}(v; p_j) = \chi(R_i) = 1,$$

where $p_1, \ldots, p_{n_i}$ are the isolated singular points of v in R_i. If there were no singular points, then the sum on the LHS would be zero, and $v \neq 0$ on C guarantees that no singular points lie on the boundary. Thus, each region R_i contains at least one singular point of v.