Hodge Decomposition Theorem

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(Extra exercise 2 of homework 8 from the linear algebra II course.)

Let V be a finite-dimensional complex inner product space. Let d be a linear operator on V such that $d^2 = 0$, and let δ be the adjoint of d (i.e., $\delta = d^*$).

(a) Prove that $\ker d\delta \subseteq \ker \delta$ and $\ker \delta d \subseteq \ker d$.

(b) Set $\Delta = d\delta + \delta d$. Show that $\ker \Delta = \ker d \cap \ker \delta$.

(c) Prove that there exists an orthogonal decomposition $V = \ker \Delta \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta$, which is called the **Hodge decomposition**.

(d) Prove that $\ker d / \operatorname{Im} d \simeq \ker \Delta$.

Proof.

(a) Suppose $x \in \ker(d\delta)$, then $\langle d\delta x, x\rangle = 0 = \langle \delta x, \delta x\rangle$, so $x \in \ker \delta$. Hence $\ker(d\delta) \subseteq \ker \delta$. Suppose $x \in \ker(\delta d)$, then $\langle \delta d x, x\rangle = \langle dx, dx\rangle = 0$, so $x \in \ker d$. Hence $\ker(\delta d) \subseteq \ker d$.

(b) ($\subseteq$): Suppose $x \in \ker \Delta$, then $(\delta d + d\delta)x = 0$, so $\langle (\delta d + d\delta)x, x\rangle = 0 = \langle \delta d x, x\rangle + \langle d\delta x, x\rangle = \langle dx, dx\rangle + \langle \delta x, \delta x\rangle$. Hence $\langle dx, dx\rangle = \langle \delta x, \delta x\rangle = 0$, hence $\delta x = dx = 0$, and $\ker D \subseteq \ker d \cap \ker \delta$.

($\supseteq$): Suppose $x \in \ker d \cap \ker \delta$, then $\Delta x = (\delta d + d\delta)x = \delta(dx) + d(\delta x) = 0$, so $x \in \ker \Delta$, and $\ker d \cap \ker \delta \subseteq \ker \Delta$.

(c) The goal is to show that $\ker \Delta$, $\operatorname{Im} d$, and $\operatorname{Im} \delta$ are mutually orthogonal and that $V = \ker \oplus \operatorname{Im} d \oplus \operatorname{Im} \delta$.

  (a) $\ker \Delta \perp \operatorname{Im} d$: Let $x \in \ker \Delta$ and $y = dx \in \operatorname{Im} d$ for some $z \in V$. Then $\langle x, y\rangle = \langle x, dz\rangle = \langle \delta x, z\rangle = 0$, since $\ker \Delta = \ker d \cap \ker \delta \subseteq \ker \delta$, from (b).

  (b) $\ker \Delta \perp \operatorname{Im} \delta$: Let $x \in \ker \Delta$ and $y = \delta z \in \operatorname{Im} \delta$ for some $z \in V$. Then $\langle x, y\rangle = \langle x, \delta z\rangle = \langle dx, y\rangle = 0$, since $\ker \Delta \subseteq \ker d$, from (b).

  (c) $\operatorname{Im} d \perp \operatorname{Im} \delta$: Let $x = du \in \operatorname{Im} d$ and $y = \delta v \in \operatorname{Im} \delta$, for some $u, v \in V$. Then $\langle x, y\rangle = \langle du, \delta v\rangle = \langle d^2 u, v\rangle = 0$, since $d^2 = 0$.

Therefore, $\ker \Delta$, $\operatorname{Im} d$, and $\operatorname{Im} \delta$ have pair-wise trivial intersection and are all finite dimensional, so it suffices to show that $\dim V = \dim(\ker \Delta) + \dim(\operatorname{Im} d) + \dim(\operatorname{Im} \delta)$. It is a property of the adjoint on finite dimensional inner product spaces that $\ker d = (\operatorname{Im} \delta)^\perp$ and $\ker \delta = (\ker d)^\perp$, so $\ker \Delta = (\operatorname{Im} \delta)^\perp \cap (\operatorname{Im} d)^\perp = (\operatorname{Im} \delta + \operatorname{Im} d)^\perp$. Then

$$\begin{aligned}\dim(\ker \Delta) &= \dim((\operatorname{Im} \delta + \operatorname{Im} d)^\perp) \\ &= \dim V - \dim \operatorname{Im} d - \dim \operatorname{Im} \delta + \dim(\operatorname{Im} d \cap \operatorname{Im} \delta) \\ &= \dim V - \dim \operatorname{Im} d - \dim \operatorname{Im} \delta,\end{aligned}$$

where the last term vanishes by orthogonality, hence the desired result.

(d) Consider the map $\phi : \ker \Delta \to \ker d / \operatorname{Im} d$, defined by $h \mapsto h + \operatorname{Im} d$. The quotient is well-defined since $\operatorname{Im} d$ is a subspace and $\operatorname{Im} d \subseteq \ker d$ by $d^2 = 0$. Furthermore, since $\ker \Delta = \ker d \cap \ker \delta$, $h \in \ker d$. Suppose $h \in \ker \phi$, then $h \in \operatorname{Im} d$, so $h \in \operatorname{Im} d \cap \ker \Delta = \{0\}$, since $\operatorname{Im} d \perp \ker \Delta$ by (c). Thus ϕ is injective. For any $h + \operatorname{Im} d \in \ker d / \operatorname{Im} d$, we can write $h = h' + d\alpha + \delta\beta$ for some $h' \in \ker \Delta$ and $\alpha, \beta \in V$ by the Hodge decomposition. Then $dh = dh' + d^2\alpha + d^2\beta = 0$, since $h \in \ker \Delta \subseteq \ker d$. Then $d\alpha + \delta\beta = h - h' \in \ker d$, $d^2\alpha + d\delta\beta = d\delta\beta = 0$, so $\beta \in \ker(d\delta) \subseteq \ker d$. Therefore $h = h' + d\alpha \in \ker \Delta \oplus \operatorname{Im} d$ and ϕ is surjective, hence it is bijective. Finally, $\phi(h+ck) = (h+ck)+\operatorname{Im} d = (h+\operatorname{Im} d)+x(k+\operatorname{Im} d) = \phi(h) + c\phi(k)$ for all $h, k \in \ker d, c \in F$, so ϕ is linear. Hence, ϕ is an isomorphism and

$$\frac{\ker d}{\operatorname{Im} d} \simeq \ker \Delta.$$

Remark. The quantity on the left hand side is the **first cohomology group** H^1, and the isomorphism implies that every cohomology class has a unique representative in $\ker \Delta$, called the **harmonic representative**.

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