Homework 10

Linear Algebra I, Fall 2024

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Exercise 1 (Section 4.3, 3). Use Cramer's rule to solve the system of linear equations

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & = & 5 \\ x_1 & - & 2x_2 & + & x_3 & = & 10 \\ 3x_1 & + & 4x_2 & - & 2x_3 & = & 0 \end{array}$$

Exercise 2 (Section 4.3, 11). A matrix $M \in M_{n\times n}(\mathbb{C})$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

Exercise 3 (Section 4.3, 13). For $M \in M_{n\times n}(\mathbb{C})$, let $\overline{M}$ be the matrix such that $(\overline{M})_{ij} = \overline{M_{ij}}$ for all i, j, where $\overline{M_{ij}}$ is the complex conjugate of M_{ij}.

(a) Prove that $\det \overline{M} = \overline{\det M}$.

(b) A matrix $Q \in M_{n\times n}(\mathbb{C})$ is called **unitary** if $QQ^* = I_n$, where $Q^* = \overline{Q^t}$. Prove that if Q is a unitary matrix, then $|\det Q| = 1$.

Exercise 4 (Section 4.3, 20). Suppose that $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det M = \det A$.

Exercise 5 (Section 4.3, 21). Prove that if $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and C are square matrices (not necessary of the same size), then $\det M = (\det A)(\det C)$.

Exercise 6 (Section 4.3, 22). Let $T : F[x]_{\leq n} \to F^{n+1}$ be the linear transformation defined by

$$T(f) = (f(c_0), f(c_1), \ldots, f(c_n)),$$

where $c_0, c_1, \ldots, c_n$ are distinct scalars in an infinite field F. Let $\mathcal{B} = \{1, x, \ldots, x^n\}$ be the standard ordered basis for $F[x]_{\leq n}$ and $\mathcal{C}$ be the standard ordered basis for F^{n+1}.

(a) Show that $M = [T]_{\mathcal{B}}^{\mathcal{C}}$ has the form

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

A matrix with this form is called a **Vandermonde matrix**.

(b) Prove that $\det M \neq 0$. (If you want to use Section 2.4, 22, you need to prove it first.)

(c) Prove that

$$\det M = \prod_{0 \leq i < j \leq n} (c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \leq i < j \leq n$.

Exercise 7 (Section 4.3, 23)**.** Let $A \in M_{n \times n}(F)$ be nonzero. For any m $(1 \leq m \leq n)$, an $m \times m$ **submatrix** is obtained by deleting any $n - m$ rows and any $n - m$ columns of A.

(a) Let k $(1 \leq k \leq n)$ denote the largest integer such that some $k \times k$ submatrix has a nonzero determinant. Prove that $\operatorname{rank} A = k$. Such k is called the **determinantal rank** of A.

(b) Conversely, suppose that $\operatorname{rank} A = k$. Prove that there exists a $k \times k$ submatrix with a nonzero determinant.

Exercise 8 (Section 4.3, 24)**.** Let $A \in M_{n \times n}(F)$ have the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix.

Exercise 9 (Section 4.3, 26(f))**.** Find the classical adjoint of the matrix

$$\begin{pmatrix} 7 & 1 & 4 \\ 6 & -3 & 0 \\ -3 & 5 & -2 \end{pmatrix}.$$

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 10. Let V be a vector space over a field F and $u_1, \dots, u_n \in V$ are linearly independent. Show that, for any $v_1, \dots, v_n \in V$, $u_1+\alpha v_1, \dots, u_n+\alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.

Exercise 11.

(a) Let $A, B, C, D \in M_{n\times n}(F)$. Prove or disprove (by giving a counterexample) that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC).$$

(b) Let $A \in M_{n\times n}(F)$ and $D \in M_{m\times m}(F)$ such that A is invertible. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

(c) Let $A, B \in M_{n\times n}(F)$. Show that

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A + B)\det(A - B).$$

(d) Let $A, B, C, D \in M_{n\times n}(F)$ be upper triangular. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC).$$

(e) Let $A, B, C, D \in M_{n\times n}(\mathbb{R})$ such that they all commute. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC).$$

Hint: Let $A' = A + tI$ for some small t. Then $t \to 0$ gives the result.

Exercise 12.

(a) Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$. Prove that $\det(I_m + AB) = \det(I_n + BA)$.

Hint: Consider the block matrix $\begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix}$.

(b) Suppose that $A \in M_{n\times n}(F)$ is invertible and $u, v \in F^n$ are column vectors. Show that

$$\det(A + uv^t) = (1 + v^t A^{-1} u)\det A.$$

This is called the **matrix determinant lemma**.