Homework 11

Linear Algebra I, Fall 2024

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Remark. In this homework, the characteristic polynomial of an $n \times n$ matrix A is defined by

$$\mathrm{ch}_A(x) := \det(xI_n - A)$$

as in the lecture.

Exercise 1 (Section 5.1, 2(d)(f)). For each of the following linear operators T on a vector space V and ordered bases $\mathcal{B}$, compute $[T]_{\mathcal{B}}$, and determine whether $\mathcal{B}$ is a basis consisting of eigenvectors of T.

(d) $V = \mathbb{R}[x]_{\leq 2}$,

$$T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2,$$

and $\mathcal{B} = \{x - x^2, -1 + x^2, -1 - x + x^2\}$.

(f) $V = M_{2\times 2}(\mathbb{R})$,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix},$$

and

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

Solution 1.

(d) Compute the following:

$$\begin{aligned} T(x - x^2) &= 4 + 4x - 4x^2 = -4(-1 - x + x^2), \\ T(-1 + x^2) &= 2 - 2x^2 = -2(-1 + x^2), \\ T(-1 - x + x^2) &= 3(x - x^2). \end{aligned}$$

Then the matrix representation of T in the ordered basis $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

Since $[T]_{\mathcal{B}}$ is not diagonal, $\mathcal{B}$ is not a basis consisting of eigenvectors of T.

(f) Compute the following:

$$\begin{aligned} T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} &= -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \\ T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \\ T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \\ T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

Then the matrix representation of T in the ordered basis $\mathcal{B}$ is $[T]_{\mathcal{B}} = \mathrm{diag}(-3, 1, 1, 1)$. Since $[T]_{\mathcal{B}}$ is diagonal, $\mathcal{B}$ is a basis consisting of eigenvectors of T.

Exercise 2 (Section 5.1, 3(d)). For the matrix

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}),$$

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution 2.

(i) By theorem 5.2, the eigenvalues are solutions to $\mathrm{ch}_A(x) = 0$. Then

$$\mathrm{ch}_A(x) = \begin{vmatrix} x-2 & 0 & 1 \\ -4 & x-1 & 4 \\ -2 & 0 & x+1 \end{vmatrix} = x(x-1)^2 = 0,$$

the eigenvalues are $\lambda = 0, 1, 1$.

(ii) In the calculations, use $v = (\alpha, \beta, \gamma)$ to denote the eigenvector corresponding to the eigenvalue in question.

$\lambda = 0$:

$$Av = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \beta = 4\alpha, \gamma = 2\alpha.$$

So the set of eigenvalues corresponding to $\lambda = 0$ is

$$S_{\lambda=0} = \{\alpha(1,4,2) \mid \alpha \in \mathbb{R} - \{0\}\}.$$

$\lambda = 1$:

$$Av = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \implies \gamma = \alpha.$$

So

$$S_{\lambda=1} = \{\alpha(1,0,1) + \beta(0,1,0) \mid \alpha, \beta \in \mathbb{R} - \{0\}\}.$$

(iii) To construct a basis $\mathcal{B}$, we take one eigenvector from $S_{\lambda=0}$ and two linearly independent eigenvectors from $S_{\lambda=1}$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Notice that the determinant of the matrix consisting of these column vectors is

$$\begin{vmatrix} 1 & 1 & -1 \\ 4 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = -2 \neq 0,$$

so $\mathcal{B}$ is linearly independent. Since $|\mathcal{B}| = 3 = \dim(\mathbb{R}^3)$, by corollary 2 (b) of theorem 1.10 $\mathcal{B}$ is a basis.

(iv) Let Q be $\mathcal{B}$ in (iii), then by Gaussian elimination we have

$$Q = \begin{pmatrix} 1 & 1 & -1 \\ 4 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix} \implies Q^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 3 & \frac{1}{2} & -\frac{5}{2} \\ -1 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

By theorem 5.1, Q diagonalise A as

$$\Lambda = Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3 (Section 5.1, 8)**.**

(a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) State and prove results analogous to (a) and (b) for matrices.

Solution 3.

(a) For both directions, we proceed by showing the contrapositive.

$(\implies)$: Suppose 0 is an eigenvalue of T, then there exists a corresponding eigenvector some $v \in V$ not equal to zero such that $T(v) = 0 \cdot v = 0$. Then $v \in \ker T \implies \{0\} \subsetneq \ker T$, so T is not one-to-one, and therefore also not invertible by theorem 2.5.

$(\impliedby)$: Suppose T is not invertible, then $\{0\} \subsetneq \ker T$, and there exists $v \in V - \{0\}$ such that $T(v) = 0 = 0 \cdot v$, so 0 is an eigenvalue of T.

(b) Let T be invertible. Then for eigenvalue λ of T, let $v \in V$ be the corresponding eigenvector. $T(v) = \lambda v$ if and only if $T^{-1}(T(v)) = v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$ if and only if $T^{-1}(v) = \lambda^{-1}v$ if and only if λ^{-1} is an eigenvalue of T. Here we have used the result of (a) (such that λ^{-1} exists if and only if T is invertible) and the fact that if T is an isomorphism, then so is T^{-1}, in particular, T^{-1} is linear.

(c) The analogous statement for matrices is:

(i) Let $A \in M_{n\times n}(F)$, then A is invertible if and only if $0 \in F^n$ is not an eigenvalue of A.

(ii) Let A be invertible, then $\lambda \in F$ is an eigenvalue of A if and only if $\lambda^{-1} \in F$ is an eigenvalue of A^{-1}.

Proof. We exploit properties of determinants.

(i) $(\implies)$: Suppose 0 is an eigenvalue of A, then there exists $0 \neq v \in F^n$ such that $Av = 0v = 0$. Then suppose A is invertible, we have $v = A^{-1}0 = 0$, contradiction, so A is not invertible.

$(\impliedby)$: Suppose A is not invertible, then $\det A = \det(A - 0 \cdot I_n) = 0$ by corollary to theorem 4.7. By theorem 5.2 0 is an eigenvalue of A.

(ii) Similar to the proof of (b), but for completeness we write it out: λ is an eigenvalue of A if and only if there exists $0 \neq v \in F^n$ such that $Av = \lambda v$ if and only if $v = \lambda A^{-1}v$ if and only if $A^{-1}v = \lambda^{-1}v$ if and only if λ^{-1} is an eigenvalue of A^{-1}.

□

Exercise 4 (Section 5.1, 12)**.**

(a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

Solution 4.

(a) Let $A \in M_{n\times n}(F)$. Notice that $\mathrm{ch}_{(Q^{-1}AQ)}(x) = \det(xI_n - Q^{-1}AQ) = \det(Q^{-1}(xI_n - A)Q) = \det(xI_n - A) = \mathrm{ch}_A(x)$, since determinant is multiplicative, and $\det(Q^{-1}) = (\det Q)^{-1}$.

(b) Let $\mathcal{B}, \mathcal{C}$ be two ordered bases for V, and $Q = [\mathsf{I}_V]_{\mathcal{C}}^{\mathcal{B}}$ be the change of coordinate matrix from basis $\mathcal{C}$ into basis $\mathcal{B}$. Let $T \in \mathcal{L}(V)$ be a linear operator on V, then $[T]_{\mathcal{C}} = Q^{-1}[T]_{\mathcal{B}}Q$. The characteristic polynomial of $[T]_{\mathcal{C}}$ is $\mathrm{ch}(x) = \det(xI_n - [T]_{\mathcal{C}}) = \det(xI_n - Q^{-1}[T]_{\mathcal{B}}Q)$. This is equal to $\det(xI_n - [T]_{\mathcal{B}})$ by (a), where $\mathcal{B}, \mathcal{C}$ are arbitrary, so the characteristic polynomial of operator T is independent of the choice of basis for V.

Exercise 5 (Section 5.1, 17)**.** Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis $\mathcal{B}$ for $M_{2\times 2}(\mathbb{R})$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

(d) Find an ordered basis $\mathcal{B}$ for $M_{n\times n}(\mathbb{R})$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix for $n > 2$.

Solution 5.

(a) Let $A \in M_{n\times n}(F)$ be an eigenvector with eigenvalue λ, then $T(A) = A^{\mathsf{T}} = \lambda A$. Taking the determinant on both sides, we get $\det(A^{\mathsf{T}}) = \det A = \det(\lambda A) = \lambda^n(\det A)$, so $\lambda = \pm 1$.

(b) By (a) the only eigenvalues are 1 and -1. Eigenvectors with eigenvalue 1 are symmetric matrices, since $A^{\mathsf{T}} = A$; eigenvectors with eigenvalue -1 are antisymmetric matrices, since $A^{\mathsf{T}} = -A$.

(c) By theorem 5.1, a linear operator is represented as a diagonal matrix in an ordered basis consisting of its eigenvectors. Therefore we try to find an independent set of order $2^2 = 4$ consisting of only symmetric and antisymmetric matrices. A natural choice is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

then

$$[T]_{\mathcal{B}} = \mathrm{diag}(1, 1, -1, 1).$$

(d) Following the same reasoning as (c), we construct a linearly independent set of order n^2 consisting of only $n \times n$ symmetric and antisymmetric matrices. Counting diagonally from left to right upward the nonzero entries, we discuss the following classes of matrices:

(i) Let the matrices with one entry only be

$$\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & 1 \end{pmatrix}.$$

(ii) Let the matrices with two entries be

$$\begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & -1 \\ \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(iii) For all the matrices with nonzero entry diagonals of size between 3 and $n-1$:

$$\begin{pmatrix} 0 & 0 & -1 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ -1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & -1 \\ \cdots & 0 & 1 & 0 \\ \cdots & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & -1 & 0 \\ \cdots & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 1 & 0 \\ \cdots & -1 & 0 & 0 \end{pmatrix}.$$

Here we use 3 as an example and list all $3 \times 2 = 6$ matrices. Notice that the negative sign appears once before each of the ones.

(iv) Finally, for matrices with nonzero entry diagonals of size n:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Lemma 1. We can evaluate the $n \times n$ determinant:

$$\Delta = \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix} = [a + (n-1)b](a-b)^{n-1}.$$

Proof. Let Δ be an $n \times n$ determinant. Then

$$\Delta = \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix} = [a + (n-1)b] \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}$$

$$= \cdots = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{vmatrix} = [a + (n-1)b](a-b)^{n-1},$$

where in the first step we added rows 2 to n to the first row and pulled the constant out of the determinant. ☐

Corollary 2.

$$\begin{vmatrix} -1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -1 \end{vmatrix} \neq 0.$$

Proof. Plug in $a = -1$ and $b = 1$ into the lemma. ☐

The matrices in classes (iii) and (iv) respectively are linearly independent by the above corollary (*), matrices with different number of nonzero terms in class (iii) are obviously linearly independent, and matrices from different classes (i) to (iv) are also obviously linearly independent. Collect the above into $\mathcal{B}$, notice that $|\mathcal{B}| = 2 \times (1+2+\cdots+(n-1)) + n = n^2$, so $\mathcal{B}$ is a basis consisting of eigenvectors. By theorem 5.1 $[T]_{\mathcal{B}}$ is diagonal, with entries in $\{1, -1\}$.

(*): Denote the matrices in $\mathcal{B}$ by $M_1, M_2, \dots, M_n$ by some preferred order (without loss of generality), and let $a_1, a_2, \dots, a_n \in F$ satisfy the equation

$$a_1 M_1 + a_2 M_2 + \cdots + a_n M_n = 0.$$

Collect the n equations of nonzero terms to get (up to some permutation)

$$\begin{cases} -a_1 + a_2 + \cdots + a_n = 0, \\ a_1 + a_2 + \cdots + a_n = 0, \\ \qquad\vdots \\ a_1 + a_2 + \cdots - a_n = 0. \end{cases}$$

So

$$\begin{pmatrix} -1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = 0.$$

Then lemma implies the matrix is invertible, and $a_1 = a_2 = \cdots = a_n = 0$. Similar reasoning can be appied to matrices in class (iii).

Exercise 6 (Section 5.1, 20)**.** Let A be an $n \times n$ matrix with characteristic polynomial

$$\operatorname{ch}_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Prove that $\operatorname{ch}_A(0) = a_0 = (-1)^n \det A$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Solution 6. By the definition of $\operatorname{ch}_A(x)$, $\operatorname{ch}_A(x) = \det(xI_n - A) = x^n + \cdots + a_1 x + a_0$. Plug in $x = 0$ to get $\det(-A) = (-1)^n(\det A) = a_0$. By corollary to theorem 5.7, A is invertible if and only if $\det A \neq 0$, if and only if $a_0 \neq 0$.

Exercise 7 (Section 5.1, 21)**.** Let A and $\operatorname{ch}_A(x)$ be as in the previous exercise.

(a) Prove that $\operatorname{ch}_A(x) = (x - A_{11})(x - A_{22}) \cdots (x - A_{nn}) + q(x)$, where $q(x)$ is a polynomial of degree at most $n - 2$.

Hint: Apply mathematical induction to n.

(b) Show that $a_{n-1} = -\operatorname{tr}(A)$.

Solution 7.

(a) For $n = 2$, we have

$$\operatorname{ch}_A(x) = \det(xI_2 - A) = (x - A_{11})(x - A_{22}) - A_{12}A_{21},$$

so $q(x) = -A_{12}A_{21}$ has degree $0 \leq 2 - 2$. By mathematical induction on n, suppose the result is true for all $n \le k-1$. Consider the case $n = k$:

$$\operatorname{ch}_A(x) = \begin{vmatrix} x - A_{11} & -A_{12} & \cdots & -A_{1k} \\ -A_{21} & x - A_{22} & \cdots & -A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{k1} & -A_{k2} & \cdots & x - A_{kk} \end{vmatrix}$$
$$= (x - A_{11}) \begin{vmatrix} x - A_{22} & -A_{23} & \cdots & -A_{2k} \\ -A_{32} & x - A_{33} & \cdots & -A_{3k} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{k2} & -A_{k3} & \cdots & x - A_{kk} \end{vmatrix} + A_{12} \begin{vmatrix} -A_{21} & -A_{23} & \cdots & -A_{2k} \\ -A_{31} & x - A_{33} & \cdots & -A_{3k} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{k1} & -A_{k3} & \cdots & x - A_{kk} \end{vmatrix}$$
$$+ \cdots + (-1)^n A_{1k} \begin{vmatrix} -A_{21} & x - A_{22} & \cdots & -A_{2,k-1} \\ -A_{31} & -A_{32} & \cdots & -A_{2,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{k1} & -A_{k2} & \cdots & -A_{k,k-1} \end{vmatrix}.$$

The first term has an $(k-1) \times (k-1)$ determinant of the desired form, so by hypothesis it is equal to $(x - A_{22}) \cdots (x - A_{kk}) + q_1(x)$, where $q_1(x)$ is of degree $k-3$. Since x only appears $k-2$ times in the latter terms, their degree are at most $k-2$. Therefore $\operatorname{ch}_A(x) = (x - A_{11}) \cdots (x - A_{kk}) + q(x)$, where the degree of $q(x)$ is at most $\max\{1 + (k-3), k-2\} = k-2$.

(b) Since the degree of $q(x)$ is at most $n-2$, the only contribution to the coefficient of x^{n-1} comes from the product $(x - A_{11}) \cdots (x - A_{nn}) = x^n - (A_{11} + \cdots + A_{nn})x^{n-1} + \cdots$. Thererfore $a_{n-1} = -\operatorname{tr} A$.

Exercise 8 (Section 5.1, 24). Use Section 5.1, 21(a) to prove Theorem 5.3:

Let $A \in M_{n \times n}(F)$.

(a) *The characteristic polynomial of A is a polynomial of degree n with leading coefficient 1. (That is, a **monic** polynomial of degree n.)*

(b) *A has at most n distinct eigenvalues.*

Solution 8.

(a) By Ex. 7 (a), $\operatorname{ch}_A(x) = x^n - (\operatorname{tr} A)x^{n-1} + \cdots$, so $\operatorname{ch}_A(x)$ is a monic polynomial in x.

(b) By theorem 5.2, λ is an eigenvalue of A if and only if $\operatorname{ch}_A(\lambda) = 0$ if and only if λ is a root of $\operatorname{ch}_A(x)$, which is a polynomial of degree n. By the Fundamental Theorem of Algebra, there are at most n distinct roots of $\operatorname{ch}_A(x)$, and therefore distinct eigenvalues.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 9. Let $A \in M_{n\times n}(F)$.

(a) Show that A is nilpotent if and only if all the eigenvalues of A are 0. (An $n \times n$ matrix A is called **nilpotent** if $A^k = O$ for some positive integer k.)

(b) What if A is idempotent? (An $n \times n$ matrix A is called **idempotent** if $A^2 = A$.)

Solution 9.

(a) ($\Longrightarrow$): Suppose A is nilpotent, then $A^k = O$ for some $k > 0$. Suppose $0 \neq v \in V$ is an eigenvector with eigenvalue λ, then $A^k v = A^{k-1}(Av) = A^{k-1}(\lambda v) = \cdots = \lambda^k v = Ov = 0$. So $\lambda = 0$.

($\Longleftarrow$): If all the eigenvalues are 0 and F is algebraically closed, then $\mathrm{ch}_A(x) = x^n$. By the Cayley-Hamilton theorem $A^n = O$, so A is nilpotent.

(b)

Claim. A matrix A is idempotent only if all of its eigenvalues are either 0 or 1. But the converse is true only if A is diagonalisable.

Proof. ($\Longrightarrow$): Suppose $A^2 = A$, then $A^2 - A = O$. For all eigenvectors $v \in V$ with eigenvalue λ, we have $(A^2 - A)v = A(\lambda v) - \lambda v = (\lambda^2 - \lambda)v = 0$, so $\lambda(\lambda - 1) = 0 \implies \lambda = 0, 1$.

For the converse, consider the counterexample

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$A^2 \neq A$ but the eigenvalues of A are 1, 1, 0. When A is diagonalisable, let $A = Q^{-1}\Lambda Q$, where Λ is diagonal with eigenvalues on the diagonal. Since the eigenvalues are all 0 or 1, $\Lambda^2 = \Lambda$, so $A^2 = (Q^{-1}\Lambda Q)(Q^{-1}\Lambda Q) = Q^{-1}\Lambda^2 Q = Q^{-1}\Lambda Q = A$. □

Exercise 10. Let $A \in M_{n\times n}(\mathbb{C})$ and let

$$\mathrm{ch}_A(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} \cdots + c_{n-1}x + c_n = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

be the characteristic polynomial of A. Note that λ_k may not be distinct.

(a) Show that

$$-kc_k = c_{k-1}\,\mathrm{tr}(A) + c_{k-2}\,\mathrm{tr}(A^2) + \cdots + c_1\,\mathrm{tr}(A^{k-1}) + \mathrm{tr}(A^k)$$

for every $1 \leq k \leq n$. (Here, we define $c_0 = 1$ and $c_k = 0$ for $k < 0$.)

Hint: Consider the classical adjoint of $xI_n - A$.

(b) Deduce that if $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^n) = 0$, then $A^n = O$.

(c) Show that

$$\mathrm{tr}(A^k) = \sum_{j=1}^{n} \lambda_j^k,$$

for every $1 \leq k \leq n$.

Remark. In fact, these statements also hold for general $A \in M_{n\times n}(F)$. As a result, the coefficients of characteristic polynomial are determined by $\mathrm{tr}(A^k)$.