# Homework 12

Linear Algebra I, Fall 2024

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**Exercise 1** (Section 5.2, 2(d)(f))**.** For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test $A$ for diagonalizability, and if $A$ is diagonalizable, find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

(d)

$$A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}.$$

(f)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Exercise 2** (Section 5.2, 3(d)(f))**.** For each of the following linear operators $T$ on a vector space $V$, test $T$ for diagonalizability, and if $T$ is diagonalizable, find a basis $\mathcal{B}$ for $V$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

(d) $V = \mathbb{R}[x]_{\leq 2}$ and $T$ is defined by $T(f(x)) = f(0) + f(1)(x + x^2)$.

(f) $V = M_{2\times 2}(\mathbb{R})$ and $T$ is defined by $T(A) = A^t$.

**Exercise 3** (Section 5.2, 13)**.** Let $A \in M_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that $A$ and $A^t$ have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. (You can directly use this fact.) For any eigenvalue $\lambda$ of $A$ and $A^t$, let $E_\lambda$ and $E'_\lambda$ denote the corresponding eigenspaces for $A$ and $A^t$, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue $\lambda$, $\dim E_\lambda = \dim E'_\lambda$.

(c) Prove that if $A$ is diagonalizable, then $A^t$ is also diagonalizable.

**Definition.** Two linear operators $T$ and $U$ on a finite-dimensional vector space $V$ are called **simultaneously diagonalizable** if there exists an ordered basis $\mathcal{B}$ for $V$ such that both $[T]_{\mathcal{B}}$ and $[U]_{\mathcal{B}}$ are diagonal matrices. Similarly, $A, B \in M_{n\times n}(F)$ are called **simultaneously diagonalizable** if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

**Exercise 4** (Section 5.2, 18)**.**

(a) Prove that if $T$ and $U$ are simultaneously diagonalizable operators, then $T$ and $U$ commute (i.e., $TU = UT$).

(b) Show that if $A$ and $B$ are simultaneously diagonalizable matrices, then $A$ and $B$ commute.

**Exercise 5** (Section 5.2, 20)**.** Let $W_1, W_2, \dots, W_k$ be subspaces of a finite-dimensional vector space $V$ such that

$$\sum_{i=1}^{k} W_i = V.$$

Prove that $V$ is the direct sum of $W_1, W_2, \dots, W_k$ if and only if

$$\dim V = \sum_{i=1}^{k} \dim W_i.$$

**Exercise 6** (Section 5.2, 21)**.** Let $V$ be a finite-dimensional vector space with a basis $\mathcal{B}$, and let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ be a partition of $\mathcal{B}$ (i.e., $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ are subsets of $\mathcal{B}$ such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ and $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$ if $i \neq j$). Prove that $V = \operatorname{span}(\mathcal{B}_1) \oplus \operatorname{span}(\mathcal{B}_2) \oplus \cdots \oplus \operatorname{span}(\mathcal{B}_k)$.

**Exercise 7** (Section 5.3, 13)**.** In 1975, the automobile industry determined that 40% of American car owners drove large cars, 20% drove intermediate-sized cars, and 40% drove small cars. A second survey in 1985 showed that 70% of the largecar owners in 1975 still owned large cars in 1985, but 30% had changed to an intermediate-sized car. Of those who owned intermediate-sized cars in 1975, 10% had switched to large cars, 70% continued to drive intermediate-sized cars, and 20% had changed to small cars in 1985. Finally, of the small-car owners in 1975, 10% owned intermediate-sized cars and 90% owned small cars in 1985. Assuming that these trends continue, determine the percentages of Americans who own cars of each size in 1995 and the corresponding eventual percentages.

**Definition.** For $A \in M_{n\times n}(\mathbb{C})$, define $e^A := \lim_{m\to\infty} B_m$, where

$$B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}$$

(if the limit exists). Thus $e^A$ is the sum of the infinite series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

and $B_m$ is the $m$th partial sum of this series. (Note the analogy with the power series

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots,$$

which is valid for all complex numbers $a$.)

**Exercise 8** (Section 5.3, 21)**.** Let $A, P, D \in M_{n\times n}(\mathbb{C})$ such that $P^{-1}AP = D$ is a diagonal matrix. Prove that $e^A = Pe^DP^{-1}$.

(There are extra exercises in the next page.)

## Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

**Exercise 9.** Let $F$ be a field and let $A, B \in M_{3\times 3}(F)$ be invertible. Suppose that $B^{-1}AB = 2A$.

(a) Find the characteristic of $F$.

(b) If $n$ is an integer not divisible by 3, prove that the matrix $A^n$ has trace 0.

(c) Prove that the characteristic polynomial of $A$ is $x^3 - a$ for some $a \in F$.

**Exercise 10.** Let $V$ be a vector space over $\mathbb{C}$ of dimension $n \geq 2$. Let $T : V \to V$ be a linear operator on $V$ with $n$ distinct eigenvalues. Prove that $V$ contains 1-dimensional subspaces $V_1, V_2, \ldots, V_n$ such that

(i) the sum

$$V = \sum_{i=1}^{n} V_i$$

is direct;

(ii) $T(V_i) \subseteq V_i + V_{i+1}$ for $1 \leq i \leq n-1$ and $T(V_n) \subseteq V_n + V_1$;

(iii) $V_i$ is not an eigenspace of $A$ for $1 \leq i \leq n$.

**Definition.** For any $A \in M_{n\times n}(\mathbb{C})$, define the **norm** of $A$ by

$$\|A\| = \max\{|A_{ij}| : 1 \leq i, j \leq n\}.$$

One can verify that it is indeed a norm, i.e., for $A, B \in M_{n\times n}(\mathbb{C})$,

(i) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = O$.

(ii) $\|cA\| = |c|\|A\|$ for any scalar $c$.

(iii) $\|A + B\| \leq \|A\| + \|B\|$.

**Exercise 11.**

(a) Let $A, B \in M_{n\times n}(\mathbb{C})$. Prove that $\|AB\| \leq n\|A\|\|B\|$.

(b) Prove that $e^A$ exists for every $A \in M_{n\times n}(\mathbb{C})$.