Homework 13

Linear Algebra I, Fall 2024

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Remark. In this homework, the characteristic polynomial of an $n \times n$ matrix A is defined by

$$\text{ch}_A(x) := \det(xI_n - A)$$

as in the lecture.

Exercise 1 (Section 5.4, 6(b)(d))**.** For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

(b) $V = \mathbb{R}[x]_{\leq 3}$, $T(f(x)) = f''(x)$, and $z = x^3$.

(d) $V = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution 1.

(b) Let $Z(z;T)$ denote the T-cyclic subspace of V generated by z. Then calculate $z = x^3$, $T(z) = 6x$, $T^2(z) = 0 = T^3(z) + \cdots$, so

$$Z(z;T) = \text{span}\left(\{x^3, 6x\}\right).$$

Since x^3 and $6x$ are linearly independent, $\{z, T(z)\} = \{x^3, 6x\}$ is an ordered basis.

(d) Calculate $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, T^2(z) = 3\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ldots$, so

$$Z(z;T) = \text{span}\left(\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, 3\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ldots, 3^m\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \ldots\right\}\right).$$

Since z and $T(z)$ are linearly independent, $\{z, T(z)\} = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right\}$ is an ordered basis.

Exercise 2 (Section 5.4, 13)**.** Let T be a linear operator on a vector space V (not necessarily finite-dimensional), let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

Solution 2.

($\Longrightarrow$): By definition of W as the span of $v, T(v), T^2(v), \ldots$, there exists some $n \in \mathbb{N}$ such that w is a linear combination of $v, T(v), \ldots, T^n(v)$. Thus $w = g(T)(v)$ for some polynomial of degree at most n, due to the linearity of T .

($\Longleftarrow$): Suppose $w = g(T)(v)$ for a polynomial g of degree n, then there exist scalars $a_0, a_1, \ldots, a_n$ such that $w = \left(a_0 + a_1T + \cdots + a_nT^n\right)(v) = a_0v + a_1T(v) + \cdots + a_nT^n(v) \in \text{span}\left(\{v, T(v), \ldots, T^n(v)\}\right) \subseteq \text{span}\left(\{v, T(v), T^2(v), \ldots\}\right) = W$.

Exercise 3 (Section 5.4, 14)**.** Prove that the polynomial $g(t)$ of the last exercise can always be chosen so that its degree is less than or equal to $\dim W$. (We view $\dim W = \infty$ if W is not finite-dimensional.)

Solution 3. Suppose W is infinite dimensional, then by assumption $\dim W = \infty$, and the claim holds trivially. So assume W is finite dimensional of dimension k. By Theorem 5.22 (a) we know $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis for W, so for all $w \in W$ there exists scalars $a_0, a_1, \ldots, a_n$ such that $w = a_0 + a_1T(v) + \cdots + a_{k-1}T^{k-1}(v) = (a_0 + a_1T + \cdots + a_{k-1}T^{k-1})$. Then $g(t) = a_0 + a_1t + \cdot + a_{k-1}t^{k-1}$ is the desired polynomial.

Exercise 4 (Section 5.4, 17)**.** Let A be an $n \times n$ matrix. Prove that

$$\dim \operatorname{span}(\{I_n, A, A^2, \ldots\}) \leq n.$$

Solution 4. By the Cayley-Hamilton Theorem for matrices, $\operatorname{ch}_A(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I_n = O$, where O is the $n \times n$ zero matrix. Then $A^n - -a_{n-1}A^{n-1} - \cdots - a_1A - A_0I_n \in \operatorname{span}\left(\{I_n, A, \ldots, A^{n-1}\}\right)$, and $A^{n+1} = -a_{n-1}A^n - \cdots - a_1A^2 - a_0A \in \operatorname{span}(\{I_n, A, \ldots, A^n\}) \subseteq \operatorname{span}\left(\{I_n, A, \ldots, A^{n-1}\}\right)$. Continuing the calculation, we see that $A^m \in \operatorname{span}\left(\{I_n, A, \ldots, A^{n-1}\}\right)$ for all $m \geq n$, so $\operatorname{span}\left(\{I_n, A, A^2, \ldots\}\right) = \operatorname{span}\left(\{I_n, A, \ldots, A^{n-1}\}\right)$, with dimension no greater than n.

Exercise 5 (Section 5.4, 19)**.** Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0.$$

Hint: Section 4.3, 24.

Solution 5.

$$\begin{aligned}\operatorname{ch}_A(x) &= \begin{pmatrix} x & 0 & \cdots & 0 & 0 & a_0 \\ -1 & x & \cdots & 0 & 0 & a_1 \\ 0 & -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x & a_{k-2} \\ 0 & 0 & \cdots & 0 & -1 & a_{k-1} \end{pmatrix} \\ &= x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0,\end{aligned}$$

by the result of Section 4.3, 24.

Exercise 6 (Section 5.4, 20)**.** Let T be a linear operator on a vector space V (not necessarily finite-dimensional), and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t)$.
Hint: Suppose that V is generated by v. Choose $g(t)$ according to Section 5.4, 13 so that $g(T)(v) = U(v)$.

Solution 6.

($\Longrightarrow$): Suppose $V = Z(v;T)$ for some $v \in V$. Since $U(v) \in V = Z(v;T)$, by Section 5.4, 13 there exists a polynomial $g(t)$ such that $g(T)(v) = U(v)$. Now suppose $TU = UT$, the goal is to show that $U(w) = g(T)(w)$ for all $w \in V$, which is true if and only if $U(T^k(v)) = g(T)(T^k(v))$ for all $k \in \mathbb{N}$, if and only if $T^k(U(v)) = g(T)(T^k(v))$, if and only if $T^k(g(T))(v) = g(T)(T^k(v))$, which is true.

($\Longleftarrow$): Suppose $U = g(T)$ for some polynomial g, then for $w \in V$, $U(w) = g(T)(w)$, and $(UT)(w) = U(T(w)) = g(T)(T(w)) = (a_0 + a_1T + \cdots + a_mT^m)(T(w)) = (a_0 + a_1T + \cdots + a_mT^{m+1})(w) = T(a_0 + a_1T + \cdots + a_mT^m)(w) = T(g(T))(w) = (TU)(w)$, for all $w \in V$. Therefore $TU = UT$.

Exercise 7 (Section 5.4, 23)**.** Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i.
Hint: Use mathematical induction on k.

Solution 7. The $k = 1$ case is obviously true. Suppose the result is true for $k = m - 1$, consider the case $k = m$. Suppose we have $u = v_1 + v_2 + \cdots + v_m \in W$, then $T(u) = T(v_1) + T(v_2) + \cdots + T(v_m) = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \in W$, since W is T-invariant by the result of Theorem 5.22.

Since $u \in W$, $\lambda_m u$ is also in W, so $T(u) - \lambda_m u = (\lambda_1 - \lambda_m)v_1 + (\lambda_2 - \lambda_m)v_2 + \cdots + (\lambda_{m-1} - \lambda_m)v_{m-1} \in W$. Since none of the $(\lambda_i - \lambda_m)$ are zero, by our induction hypothesis all of $(\lambda_i - \lambda_m)v_i$ for $1 \leq i \leq m-1$ are in W, and so $v_i \in W$ for $1 \leq i \leq m-1$. Finally, $v_m = u - v_1 - \cdots - v_{m-1} \in W$, and we are done.

Exercise 8 (Section 5.4, 25)**.**

(a) Prove that if T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $UT = TU$, then T and U are simultaneously diagonalizable.
Hint: For any eigenvalue λ of T, show that E_λ is U-invariant, and apply Section 5.4, 24 to obtain a basis for E_λ of eigenvectors of U.

(b) State and prove a matrix version of (a).

Solution 8.

(a) First we begin with the result of Section 5.4, 24 as a lemma.

Lemma 1 (Section 5.4, 24)**.** If T is a diagonalisable linear operator, then the restriction of T to any nontrivial T-invariant subspace of V is also diagonalisable.

Proof. Let $T \in \mathcal{L}(V)$ be diagonalisable, $\{0\} \subsetneq W \subseteq V$ a T-invariant subspace of V, and E_{λ_i} the eigenspace corresponding to eigenvalue λ_i. Furthermore, let $W_i = E_{\lambda_i} \cap W$ be the eigenspace of $T|_W$ corresponding to λ_i.

For each of W_i, we can find a basis $\mathcal{B}_i$ for it. We claim that $\mathcal{B} = \bigcup_i \mathcal{B}_i$ is a basis for W. By Theorem 5.8, since $\mathcal{B}_i$ is a linearly independent subset of W_i, $\mathcal{B}$ is a linearly independent subset of W. Furthermore, since T is diagonalisable, by Theorem 5.1 every vector in V, and therefore in W, is in the span of the set of the eigenvectors corresponding to distinct eigenvalues. Then by Section 5.4, 23, these eigenvectors must themselves be in W.

Then $W = \text{span}(\mathcal{B})$, so $\mathcal{B}$ is a basis for W consisting of eigenvectors. By Theorem 5.1, $[T|_W]_{\mathcal{B}}$ is diagonal. $\square$

Suppose T, U are diagonalisable and $TU = UT$. For some eigenvalue λ of T, let E_λ be its eigenspace and $v_i \in E_\lambda$ not necessarily an eigenvector. Then $T(U(v)) = (TU)(v) = (UT)(v) = \lambda U(v)$, so $U(v) \in E_\lambda$

Apply the above to each E_{λ_i}, we see that all of them are U-invariant, so we may Lemma 1 to each E_{λ_i}, giving us a basis $\mathcal{B}_i$ such that $[U_i]_{\mathcal{B}_i}$ is diagonal. Take $\mathcal{B} = \bigcup_i \mathcal{B}_i$, where the union is taken by placing each of $\mathcal{B}_1, \mathcal{B}_2, \ldots$ in order. Then $[T]_{\mathcal{B}}$ and $[U]_{\mathcal{B}}$ are both diagonal.

(b) We state an analogous result for matrices $A, B \in M_{n\times n}(F)$: If A, B are diagonalisable and $AB = BA$, then A and B are simultaneously diagonalisable.

Proof. Apply the result of (a) to the linear operators L_A and L_B and we are done. □

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 9. Let $A \in M_{2\times 2}(\mathbb{C})$ with distinct eigenvalues, and let

$$S = \left\{ B \in M_{2\times 2}(\mathbb{C}) : \begin{pmatrix} A & B \\ O & A \end{pmatrix} \text{ is diagonalizable} \right\}$$

be a subset of $M_{2\times 2}(\mathbb{C})$. Show that S is a 2-dimensional subspace of $M_{2\times 2}(\mathbb{C})$.

Exercise 10. Fix an integer $n > 1$. For $A, B \in M_{n\times n}(\mathbb{C})$, define

$$[A, B] := AB - BA.$$

Let $\ker M$ denote the kernel of the linear operator given by left multiplication by $M \in M_{n\times n}(\mathbb{C})$, that is,

$$\ker M = \{A \in M_{n\times n}(\mathbb{C}) : MA = O\}.$$

Now, fix $A, B \in M_{n\times n}(\mathbb{C})$ and define

$$\mathcal{N} = \bigcap_{k=1}^{n} \bigcap_{\ell=1}^{n} \ker[A^k, B^\ell].$$

Show that A and B have a common eigenvector if and only if $\mathcal{N}$ contains a nonzero vector.