Homework 3

Linear Algebra I, Fall 2024

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Exercise 1 (Section 1.5, 15)**.** Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \dots, u_k\})$ for some k $(1 \le k < n)$.

Solution 1. We prove the two directions of the *iff* condition.

($\implies$): Suppose S is linearly dependent, then there exists some $(a_1, \dots, a_n) \in F^n$ not all zero such that

$$a_1 u_1 + \cdots + a_n u_n = 0.$$

The sequence $(a_1, a_2, \dots, a_n)$ has non-zero terms. We call the non-zero coefficient with largest index a_l, where l satisfies $1 \le l < n$. Consider the two cases:

1. $l = 1$: $a_1 u_1 + 0 + \cdots + 0 = 0, a_1 \neq 0$. Then it must be true that $u_1 = 0$.
2. $1 < l < n$: We can write

$$a_1 u_1 + \cdots a_{l-1} u_{l-1} + a_l u_l = 0,$$

   Then

$$u_l = -\left(\frac{a_1}{a_l}\right) u_1 - \left(\frac{a_2}{a_l}\right) u_2 - \cdots - \left(\frac{a_{l-1}}{a_l}\right) u_{l-1} \in \operatorname{span}\left(\{u_1, \dots, u_{l-1}\}\right).$$

   Let $l = k+1$ to get the desired result.

($\impliedby$): The case $u_1 = 0$ is trivial, since the n-tuple $(a_1, 0, \dots, 0) \in F^n, a_1 \neq 0$ works.

Consider the case $u_{k+1} \in \operatorname{span}\left(\{u_1, \dots, u_k\}\right)$: we can write u_{k+1} in the $\{u_1, \dots, u_k\}$ basis as

$$u_{k+1} = b_1 u_1 + \cdots b_k u_k.$$

Then we can write the non-trivial linear combination

$$(-b_1)u_1 + (-b_2)u_2 + \cdots + (-b_k)u_k + u_{k+1} + 0 \cdot u_{k+2} + \cdots + 0 \cdot u_n = 0,$$

so S is linearly dependent.

Exercise 2 (Section 1.5, 16)**.** Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution 2. We prove the two directions of the *iff* condition.

($\implies$): Suppose by way of contradiction that S is linearly independent and there exists a finite subset, say W, of S with dimension n that is linearly dependent. Then there is some tuple $(a_1, \dots, a_n) \in F^n$ not all zero that satisfies

$$a_1 w_1 + \cdots + a_n w_n = 0,\ w_1, \dots, w_n \in W.$$

Then there is the nontrivial representation of $0 \in S$ using $w_1, \dots, w_n$:

$$a_1 w_1 + a_2 w_2 + \cdots + a_n w_n = 0.$$

Thus the contradiction.

($\Longleftarrow$): This is trivial if S is finite, since S is its own finite subset. So assume S is an infinite set. From the definition of linear dependence, an infinite set of vectors is *linearly dependent* if it contains a subset that is linearly dependent. So, by negation, an infinite set of vectors is linearly independent if every (finite) subset is linearly independent. Therefore, this direction is also trivial for the infinite case.

Remark. The ($\Longrightarrow$) direction can be trivially shown by invoking theorem 1.6: If V is a vector space, $S_1 \subseteq S_2 \subseteq V$ its subsets, and S_2 is linearly independent, then S_1 is linearly independent.

Exercise 3 (Section 1.6, 13)**.** The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Solution 3. Simplifying the equation gives

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

So $x_1 = x_2 = x_3$. A spanning set for this subspace is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

such that $\operatorname{span}(\mathcal{B}) \subset \mathbb{R}^3$. This is a basis since a non-zero singleton is linearly independent.

Exercise 4 (Section 1.6, 14)**.** Find bases for the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

Solution 4.

1. Substitute in the relation to get

   $$W_1 = \{(a_3 + a_4, a_2, a_3, a_4, a_5) \in F^5\}.$$

   Notice that

   $$(a_3 + a_4, a_2, a_3, a_4, a_5) = a_2(0, 1, 0, 0, 0) + a_3(1, 0, 1, 0, 0) + a_4(1, 0, 0, 1, 0) + (0, 0, 0, 0, 1).$$

   Write

   $$\mathcal{B}_1 = \{(0, 1, 0, 0, 0), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1)\},$$

   checking the vectors for linear independence shows that $\mathcal{B}$ is indeed linearly independent, so $\mathcal{B}$ is a basis for W_1, and

   $$\dim W_1 = |\mathcal{B}_1| = 4.$$

2. Substitute in the relation to get

$$W_2 = \{(a_1, 0, 0, 0, -a_1) \in F^5\}.$$

Notice that

$$(a_1, 0, 0, 0, -a_1) = a_1(1, 0, 0, 0, -1) + a_2 \cdot (0, 1, 1, 1, 0).$$

Write

$$\mathcal{B}_2 = \{(1, 0, 0, 0, -1), (0, 1, 1, 1, 0)\},$$

checking the vectors for linear independence shows that $\mathcal{B}$ is indeed linearly independent, so $\mathcal{B}$ is a basis for W_2, and

$$\dim W_2 = |\mathcal{B}_2| = 2.$$

Exercise 5 (Section 1.6, 20)**.** Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)

(b) Prove that S contains at least n vectors.

Solution 5.

(a) To prove that there exists $\mathcal{B} \subseteq S$ such that $\mathcal{B}$ is a basis for V, first take a basis $\mathcal{B}' \subseteq V$: $\mathcal{B}' = \{\beta_1, \ldots, \beta_n\}$.

Then for $\alpha_{ij} \in F, s_{ij} \in S, 1 \leq i \leq n, 1 \leq j \leq n_i$, we can write the b_i's as

$$\begin{cases} b_1 = \alpha_{11}s_{11} + \cdots + \alpha_{1n_1}s_{1n_1}, \\ \vdots \\ b_n = \alpha_{n1} + \cdots + \alpha_{nn_n}. \end{cases}$$

This implies $\operatorname{span}\mathcal{B}' \subseteq \operatorname{span}(\{s_{ij}\}) \subseteq V$. But $\operatorname{span}\mathcal{B}' = V$, so $\operatorname{span}(\{s_{ij}\}) = V$.

The set $\{s_{ij}\}$ is finite, so we can use theorem 1.10 (Steinitz exchange lemma / replacement theorem) to extract a basis from $\{s_{ij}\}$. Call it $\mathcal{B}$, and $\mathcal{B} \subseteq S$ by construction.

(b) Suppose $S \subset V$ has $m < n$ elements and spans V, then we can choose the subset $\overline{S} \subset S$ that is linearly independent, so that $\operatorname{span}(\overline{S}) = \operatorname{span}(S) = \dim(V)$.

Then $\overline{S}$ is a basis for V, which is impossible since $\left|\overline{S}\right| \leq |S| = m < \dim(V)$. Thus, $|S| \geq n$.

Remark. An algorithmic procedure to "extract" basis vectors from S may not work when S is infinite, since the procedure of searching through every vector in S cannot terminate in finite time.

Exercise 6 (Section 1.6, 22)**.** Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(W_1 \cap W_2) = \dim W_1$.

Solution 6. Claim:

$$\dim(W_1 \cap W_2) = \dim(W_1) \iff W_1 \subset W_2.$$

($\implies$): notice that $W_1 \cap W_2 \subset W_1$ is a subspace of W_1, since it is a subset, and

1. Both W_1 and W_2 are subspaces, so they contain the zero vector. Therefore their intersection contains 0.

2. $u, v \in W_1 \cap W_2 \implies u + v \in W_1 \cap W_2$, since W_1 and W_2 are subspaces.

3. $u \in W_1 \cap W_2 \implies cu \in W_1 \cap W_2 \forall c \in F$, again since W_1 and W_2 are subspaces.

Lemma 1. If a subspace has the same dimension as the vector space, then they are equal.

Proof. Assume $W \subset V$ is a subspace of V with basis β , where $\dim V = n$. Let the dimension of W also be n. Then for some vector $u \in V$, either $u \in \text{span}(W)$ or is not. If not, then the set $\{\beta, u\}$ is a basis with $n + 1$ elements, which is a contradiction. □

By our lemma, $\dim(W_1 \cap W_2) = \dim(W_1)$ implies $W_1 \cap W_2 = W_1$. Thus, $W_1 \subset W_2$.

$(\Longleftarrow)$: $W_1 \subseteq W_2 \implies W_1 \cap W_2 = W_1 \implies \dim(W_1 \cap W_2) = \dim(W_1)$.

Exercise 7 (Section 1.6, 29)**.** (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Hint: Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ for W_1 and to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$ for W_2.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim V = \dim W_1 + \dim W_2$.

Solution 7.

(a) Let $\beta = \{u_1, \ldots, u_k\}$ be a basis for $W_1 \cap W_2$. Since $W_1 \cap W_2$ is a subset of W_1 and W_2, we can extend β to form bases for W_1 and W_2:

$\dim(W_1) = m$, so let $\beta_1 = \beta \cup \{v_{k+1}, \ldots, v_m\}$ be a basis for W_1. Similarly, $\dim(W_2) = n$, so let $\beta_2 = \beta \cup \{w_{k+1}, \ldots, w_n\}$ be a basis for W_2.

The β part is identical for the bases of W_1 and W_2, so we can construct the following spanning set of $W_1 + W_2$: $\gamma = \{u_1, \ldots, u_k, v_{k+1}, \ldots, v_m, w_{k+1}, \ldots, w_n\}$. Now we only have to show that γ is linearly independent.

This can be shown as follows: $\{c_1, \ldots, c_k, a_{k+1}, \ldots a_m, b_{k+1}, \ldots, b_n\} \in F^{k+(m-k)+(n-k)}$ is a set of scalars, consider the linear combination

$$a_{k+1}v_{k+1} + \cdots + a_m v_m + b_{k+1}w_{k+1} + \cdots b_n w_n + c_1 u_1 + \cdots c_k u_k = 0.$$

This can also be written as

$$a_{k+1}v_{k+1} + \cdots + a_m v_m = -b_{k+1}w_{k+1} - \cdots b_n w_n - c_1 u_1 - \cdots c_k u_k,$$

so $a_{k+1}v_{k+1} + \cdots + a_m v_m \in W_2$. But the LHS is in W_1, so it must be that $a_{k+1}v_{k+1} + \cdots + a_m v_m \in W_2 \in W_1 \cap W_2$.

Then we can write

$$a_{k+1}v_{k+1} + \cdots + a_m v_m = d_1 u_1 + \cdots + d_k u_k$$

for some $\{d_1, \ldots, d_k\} \in F^k$. By the linear independence of W_1, we have $a_{k+1} = \cdots = a_m = 0$. Going back to the original equation, we are now left with

$$b_{k+1}w_{k+1} + \cdots b_n w_n + c_1 u_1 + \cdots c_k u_k = 0,$$

and by linear independence of W_2 we have $b_{k+1} = \cdots = b_n = c_1 = \cdots = c_k = 0$, thus γ is linearly independent. Now we can safely write

$$\dim(W_1 + W_2) = n + m - k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2),$$

and $\dim(W_1 + W_2)$ is therefore finite.

(b) We prove the two directions:

($\Longrightarrow$): Suppose $V = W_1 \oplus W_2$, then by definition $V = W_1 + W_2$ and $W_1 \cap W_2 = \varnothing$. Then from (a) we have

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + 0 = \dim(W_1) + \dim(W_2).$$

($\Longleftarrow$): By (a), we have $\dim(W_1 \cap W_2) = 0$. So $W_1 \cap W_2 = \varnothing$, which, along with the assumption $V = W_1 + W_2$, implies $W = W_1 \oplus W_2$.

Exercise 8 (Section 1.6, 31)**.** Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Solution 8. The solution uses result from Ex. 7 (a).

(a) Suppose $\dim(W_1 \cap W_2) = p > n$, then there is a basis $\overline{\beta} = \{u_1, \ldots, u_p\}$ for $W_1 \cap W_2$.

Since $W_1 \cap W_2$ is a subset of W_1, this implies there is a linearly independent set of $p > n$ elements in W_1, contradicting the fact that $\dim(W_1) = n$. Thus, $\dim(W_1 \cap W_2) \leq n$.

(b) From Ex. 7 (a) we have the result

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = m + n - \alpha,$$

for some $\alpha \geq 0$, so $\dim(W_1 + W_2) \leq m + n$.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 9. Suppose that V is a finite-dimensional vector space and U is a subspace of V. Show that there exists a subspace W of V such that $V = U \oplus W$.
In this case, we call W a **complementary subspace** of U in V.

Solution 9. This exercise shows the existence of a complementary subspace.

If U is a subspace in V, then the basis of U, which we call $\mathcal{B}_U$ is in V. Following the method in solution 7 (a), we extend $\mathcal{B}_U$ to a basis for V, $\mathcal{B}_V$, by adding in linearly independent vectors.

Since the vectors in $\mathcal{B}_V$ are linearly independent, we have

$$\operatorname{span}(\mathcal{B}_U \backslash \mathcal{B}_U) \cap \operatorname{span}(\mathcal{B}_U) = \{0\}.$$

This is true because any vector $v \in V$ can be represented by a linear combination of $\{u_1, \dots, u_m, w_1, \dots, w_{n-m}\}$, where $u_1, \dots, u_m \in U$ and $w_1, \dots, w_{n-m} \in V \backslash U$.

Let $W = \operatorname{span}(B_V \backslash B_U)$, then we have $V = U \oplus W$, as desired.

Exercise 10. Let $\mathbb{R}^+$ be the set of all positive real numbers. One can verify that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$x \boxplus y = xy, \quad x, y \in \mathbb{R}^+$$

and the scalar multiplication

$$a \boxtimes x = x^a, \quad x \in \mathbb{R}^+, a \in \mathbb{R}.$$

Find a basis and the dimension of this vector space.

Solution 10. We guess that the basis consists of a single element not equal to 1. Let it be 3, so $\mathcal{B} = \{3\}$. We want to show that any element $x \in \mathbb{R}^+$ can be written as $0 < x = c \boxtimes 3 = 3^c, c \in \mathbb{R}$, but then c is simply $\log_3(x)$. So $\mathcal{B} = \{3\}$, $\dim(\mathbb{R}^+) = 1$.

Remark. The map $\log_3 : \mathbb{R}^+ \to \mathbb{R}$ is an isomorphism, since it is linear (with respect to addition anf multiplication as defined above), onto, and one-to-one. So this also proves that $\mathbb{R}^+ \cong \mathbb{R}$.

Exercise 11. Let $n \in \mathbb{N}$. For $a = 0, 1, \dots, n$, define

$$P_{n,a} = (x + a)(x + a + 1) \cdots (x + a + n - 1)$$

to be a polynomial in $\mathbb{R}[x]_{\leq n}$.

(a) Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P_{n,k} = (-1)^n n!.$$

Hint: First show that $P_{n,k} - P_{n,k-1} = n \cdot P_{n-1,k}$.

(b) Show that $\{P_{n,a} : 0 \leq a \leq n\}$ is a basis of $\mathbb{R}[x]_{\leq n}$.

Hint: Show that

$$\{1, x, x(x+1), x(x+1)(x+2), \dots, x(x+1) \cdots (x+n)\} \subseteq \operatorname{span}(\{P_{n,a} : 0 \leq a \leq n\}).$$