Homework 4

Linear Algebra I, Fall 2024

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Exercise 1 (Section 2.1, 14)**.** Let V and W be vector spaces and $T : V \to W$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\mathcal{B}) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Solution 1.

(a) Suppose $S \subseteq V$ is a subset.

($\implies$) We assume T is one-to-one, and that the set $T(S) \subseteq W$ is linear independent, so that every finite subset is linearly independent (see HW3). We want to show that S is linear independent.

Take an arbitrary (finite) subset $\{T(s_1), T(s_2), \ldots, T(s_k)\} \subseteq T(S)$, where $s_1, \ldots, s_k \in S$. Then

$$a_1T_1(s_1) + \cdots + a_kT(s_k) = 0 \implies a_1 = \cdots = a_k = 0.$$

But

$$a_1T_1(s_1) + \cdots + a_kT(s_k) = T(a_1s_1 + \cdots a_ks_k) = 0,$$

so $a_1s_1 + \cdots a_ks_k \in \ker T = \{0\}$, by theorem 2.4. Then $a_1s_1 + \cdots a_ks_k = 0 \implies a_1 = \cdots = a_k = 0$, so $\{s_1, \ldots, s_k\}$ is linearly independent. Thus S is linearly independent.

($\impliedby$) We shall show that $\ker T = \{0\}$. Take a vector $x \in \ker T$, suppose the subset $S \subseteq V$ is linearly independent, and so is $T(S) \subseteq W$. We can write x as

$$x = a_1s_1 + \cdots a_ks_k,\ a_1, \ldots, a_k \in F,\ s_1, \ldots, s_k \in S.$$

Then $T(x) = 0 \implies T(a_1s_1 + \cdots + a_ks_k) = a_1T(s_1) + \cdots + a_kT(s_k) = 0$. Linear independence of $T(S)$ implies that $a_1 = \cdots = a_k = 0$, so $x = 0$. Therefoe $\ker T = \{0\}$, and T is one-to-one by theorem 2.4.

(b) We suppose $S \subseteq V$ is a subset, T is one-to-one, and proceed to a proof by showing the contrapositive in both directions.

($\implies$) Suppose that $T(S)$ is not linearly independent, so that for a finite subset $\overline{S} = \{s_1, \ldots, s_k\} \subseteq S$, we have a finite subset $\overline{T} = \{T(s_1), \ldots, T(s_k)\} \subseteq T(S)$ such that there exists $a_1, \ldots, a_k \in F$ not all zero, $a_1T(s_1) + \cdots a_kT(s_k) = 0$. But then

$$a_1T(s_1) + \cdots a_kT(s_k) = T(a_1s_1 + \cdots + a_ks_k) = 0 = T(0) \implies a_1s_1 + \cdots + a_ks_k = 0,$$

since T is assumed to be one-to-one. Therefore $\overline{S}$ is not linearly independent, for all finite subsets $\overline{S}$ of S.

($\impliedby$) Suppose to the contrary that there exists $a_1, \ldots, a_k \in F$ not all zero such that for a finite subset $\overline{S} = \{s_1, \ldots, s_k\} \subseteq S$, $a_1s_1 + \cdots + a_ks_k = 0$. Then $T(a_1s_1 + \cdots + a_ks_k) = T(0) = 0 = a_1T(s_1) + \cdots + a_kT(s_k)$, so $T(S)$ is not linearly independent.

(c) To show that $T(\mathcal{B})$ is a basis for W, we shall prove the following:

(i) $T(\mathcal{B})$ is a spanning set of W: Since T is onto , by definition $\operatorname{im} T = W$. By theorem 2.2, $\operatorname{im} T = \operatorname{span}(T(\mathcal{B}))$, so $\operatorname{span}(T(\mathcal{B})) = W$.

(ii) $T(\mathcal{B})$ is linearly independent: For scalars $b_1, \dots, b_n \in F$ such that $b_1 T(v_1) + \cdots + b_n T(v_n) = 0$, we then have

$$b_1 T(v_1) + \cdots + b_n T(v_n) = T(b_1 v_1 + \cdots + a_n v_n) = 0 = T(0).$$

Then $b_1 v_1 + \cdots + b_n v_n = 0$, since T is one-to-one, and $b_1 = \cdots = b_n = 0$ by linear independence of $\mathcal{B}$.

Therefore $\mathcal{B}$ is a basis for W.

Exercise 2 (Section 2.1, 21)**.** Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U : V \to V$ by

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) \quad \text{and} \quad U(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Solution 2.

(a) Let $(a_1, a_2, \dots), (b_1, b_2, \dots) \in V$, $c \in F$.

(i)

$$\begin{aligned} T(c(a_1, a_2, \dots) + (b_1, b_2, \dots)) &= T(ca_1 + b_1, ca_2 + b_2, \dots) \\ &= (ca_2 + b_2, \dots) = c(a_2, a_3, \dots) + (b_2, b_3, \dots) \\ &= cT(a_1, a_2, \dots) + T(b_1, b_2, \dots). \end{aligned}$$

(ii)

$$\begin{aligned} U(c(a_1, a_2, \dots) + (b_1, b_2, \dots)) &= U(ca_1 + b_1, ca_2 + b_2, \dots) \\ &= (0, ca_1 + b_1, \dots) = c(0, a_1, a_2, \dots) + (0, b_1, b_2, \dots) \\ &= cU(a_1, a_2, \dots) + U(b_1, b_2, \dots). \end{aligned}$$

Therefore T, U are linear.

(b) Since $0 \in V$, for all sequences $(a_2, a_3, \dots) \in V$ we can find the sequence $(0, a_2, a_3, \dots) \in V$ such that $T(0, a_2, a_3, \dots) = (a_2, a_3, \dots)$, so T is onto. But for $a_1 \neq \overline{a}_1$, we have $T(a_1, a_2, \dots) = (a_2, a_3, \dots) = T(\overline{a}_1, a_2, \dots)$, so T is not one-to-one.

(c) Suppose for $a_1, a_2, \dots, b_1, b_2, \dots \in F$ we have $U(a_1, a_2, \dots) = U(b_1, b_2, \dots)$, then

$$(0, a_1, a_2, \dots) = (0, b_1, b_2, \dots) \implies (a_1, a_2, \dots) = (b_1, b_2, \dots).$$

Thus U is one-to-one. But for $a \neq 0$, $(a, a_1, a_2, \dots) \in V$ is not in $U(V)$, so U is not onto.

Exercise 3 (Section 2.1, 22)**.** Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$. Can you generalize this result for $T : F^n \to F$? State and prove an analogous result for $T : F^n \to F^m$.

Solution 3. Note that

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. Let

$$a = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we see that $a, b, c \in \mathbb{R}$. Then by linearity of T, we have

$$\begin{aligned} T(x, y, x) = T \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T \left(x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= xT \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + yT \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + zT \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= ax + by + cz. \end{aligned}$$

We claim that the analogous case for $T : F^n \to F^m$ is simply

$$T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix},$$

where $(x_1, \ldots, x_n) \in F^n, \; a_{11}, \ldots, a_{mn} \in F$. We prove this by noticing that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ is a basis for F^n, where we simply write 1 for 1_F. Let

$$T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, T \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, T \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1n} \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \in F^m.$$

Then

$$\begin{aligned} T(x_1, \ldots, x_n) = T \left(x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right) &= x_1 T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n T \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}. \end{aligned}$$

Remark. This means we could write a linear transformation as a matrix with respect to some basis.

Definition. Let V be a vector space, and let $T : V \to V$ be linear. A subspace W of V is said to be **T-invariant** if $T(x) \in W$ for every $x \in W$, that is, $T(W) \subseteq W$. If W is T-invariant, we define the **restriction of T on W** to be the function $T|_W : W \to W$ defined by $T|_W(x) = T(x)$ for all $x \in W$.

Exercise 4 (Section 2.1, 28)**.** Let V be a vector space, and let $T : V \to V$ be linear. Prove that the subspaces $\{0\}$, V, $\operatorname{image} T$, and $\ker T$ are all T-invariant.

Solution 4.

1. $T(0) = 0 \in \{0\}$, so $\{0\}$ is T-invariant.
2. Since T is a map from V to V, by definition $T(V) \subseteq V$, so V is T-invariant.
3. By theorem 2.1, $\operatorname{im} T \equiv \{T(x)|x \in V\}$ is a subspace of V, so $T(w) \in \operatorname{im} T, \forall w \in \operatorname{im} T \subseteq V$ by definition of $\operatorname{im} T$. So $T(\operatorname{im} T) \subseteq \operatorname{im} T$, $\operatorname{im} T$ is T-invariant.
4. $T(x) = 0 \in \ker T$ for all $x \in \ker T$, so $T(\ker T) \subseteq \ker T$, $\ker T$ is T-invariant.

Exercise 5 (Section 2.1, 29)**.** If W is a T-invariant subspace of a vector space V and $T : V \to V$ is linear, prove that $T|_W$ is linear.

Solution 5. Take $x, y \in W$ and $c \in F$, then $x + cy \in W$ since W is a subspace. Then

$$T|_W(x + cy) = T(x + cy) = T(x) + cT(y) = T|_W(x) + cT|_W(y).$$

Exercise 6 (Section 2.1, 32)**.** Suppose that W is a T-invariant subspace of a vector space V and that $T : V \to V$ is linear. Prove that $\ker T|_W = \ker T \cap W$ and $\operatorname{image} T|_W = T(W)$.

Solution 6. We prove that (1) $\ker(T|_W) = \ker T \cap W$ and (2) $\operatorname{im}(T|_W) = T(W)$.

(1) Suppose $x \in \ker(T|_W)$, then $T|_W(x) = 0 = T(x)$, so $x \in \ker T$. Also, $x \in W$ by definition of domain of $T|_W$. Then $x \in \ker T \cap W \implies \ker(T|_W) \subseteq \ker T \cap W$. Now suppose $x \in \ker T \cap W$, then $T|_W(x) = T(x) = 0 \implies x \in \ker T \implies \ker T \cap W \subseteq \ker(T|_W)$. Thus $\ker(T|_W) = \ker T \cap W$.

(2) In W, we have $T|_W(x) = T(x)$, so $\operatorname{im}(T|_W) \equiv \{T|_W(x)|x \in W\} = \{T(x)|x \in W\} = T(W)$.

Exercise 7 (Section 2.1, 37)**.** A function $T : V \to W$ between vector spaces V and W is called **additive** if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

Solution 7. We want to show that $\forall u, v \in V, c \in \mathbb{Q}$, we have $T(u + cv) = T(u) + cT(v)$. By the additive property we already have $T(u + cv) = T(u) + T(cv)$, so we only have to show that $T(cv) = cT(v)$ for all $v \in V, c \in \mathbb{Q}$.

The case $c = 0$ is trivial, so consider the case $c > 0$. We can let $c = p/q$, where $p, q \in \mathbb{N}$.

(1) The value of $T(\frac{v}{q})$ can be define:

$$T\left(q \cdot \left(\frac{v}{q}\right)\right) = T\left(\frac{v}{q} + \cdots + \frac{v}{q}\right) = T\left(\frac{v}{q}\right) + \cdots + T\left(\frac{v}{q}\right)$$
$$\implies T\left(\frac{v}{q}\right) = \frac{1}{q}T(v).$$

(2) Then

$$T\left(\left(\frac{p}{q}\right)v\right) = T\left(p\left(\frac{v}{q}\right)\right) = T\left(\frac{v}{q}\right) + \cdots + T\left(\frac{v}{q}\right) = pT\left(\frac{v}{q}\right) = \frac{p}{q}T(v),$$

so $T(cv) = cT(v)$ for $c \geq 0$.

The case $c < 0$ can be included simply by noticing that

$$T\left(cv + |c|v\right) = T(0) = 0 = T(cv) + T(|c|v).$$

$$T(cv) = -T(|c|v).$$

Therefore, for all $c \in \mathbb{Q}$ there is $T(cv) = cT(v)$, and $T(u + cv) = T(u) + cT(v)$, as desired.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 8. Let V be a vector space over F and let $T : V \to V$ be a linear transformation. Suppose that every subspace of V is T-invariant. Show that T is a scalar multiple of the identity transformation I_V.

Exercise 9 (Section 1.3, 31)**.** Let W be a subspace of a vector space V over a field F. For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is called the **coset of W containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of F can be defined in the collection

$$S = \{v + W : v \in V\}$$

of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$.

(c) Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** and is denoted by V/W.

Exercise 10 (Section 1.6, 35, modified)**.** Let W be a subspace of a finite-dimensional vector space V, and consider the basis $u_1, u_2, \ldots, u_k$ for W. Let $u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n$ be an extension of this basis to a basis for V.

(a) Prove that $\{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$ is a basis for V/W.

(b) Derive the formula $\dim(V/W) = \dim V - \dim W$.