Homework 5

Linear Algebra (I), Fall 2024

Deadline: 10/9 (Wed.) 12:10

Exercise 1 (Section 2.2, 10)**.** Let V be a vector space with the ordered basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6, there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[T]_{\mathcal{B}}$.

Solution 1. Under the basis $\mathcal{B}$, the linear transformation T can be represented as an $n \times n$ matrix, with components a_{ij}. The coefficients $a + ij$ should satisfy

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i = v_j + v_{j-1}.$$

By observation, we find that $a_{ij} = 1$ if $i = j$ or $i = j - 1$, and $a_{ij} = 0$ otherwise:

$$[T(v_1)]_{\mathcal{B}} = (1\ 0\ \cdots\ 0)^{\intercal},$$
$$[T(v_n)]_{\mathcal{B}} = (0\ 0\ \cdots\ 1)^{\intercal},$$
$$[T(v_j)]_{\mathcal{B}} = (0\ \cdots\ 0\ 1\ 1\ 0\ \cdots 0)^{\intercal},\ 1 < j < n.$$

The final matrix is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}.$$

Exercise 2 (Section 2.2, 11)**.** Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V having dimension k. Show that there is a basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}}$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n - k) \times k$ zero matrix.

Solution 2. The cases $k = 0$ and $k = n$ are trivial, since then there is no restriction on the form of the matrix, and the matrix must exist.

Consider $1 < k < n$, let $\overline{\mathcal{B}} = \{v_1, \ldots, v_k\}$ be a basis of W. Since W is T-invariant, $T(W) \subseteq W \implies T(v_j) \in W$, for $1 \le j \le k$, and we can write

$$T(v_j) = \sum_{i=1}^{k} a_{ij} v_i.$$

Extend $\overline{\mathcal{B}}$ to a basis for V, $\mathcal{B} = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$, by the replacement theorem. Then for $1 \le j \le k$, $T(v_j)$ is

$$T(v_j) = a_{1j} v_1 + \cdots + a_{kj} v_k + 0 \cdot v_{k+1} + \cdots + 0 \cdot v_n.$$

In coordinate vector form this is $[T]_{\mathcal{B}} = (a_{1j}\ \cdots a_{kj}\ 0\ \cdots\ 0)^{\intercal}$, so

$$[T]_{\mathcal{B}} = [T(v_1)\ \cdots\ T(v_k)\ T(v_{k+1}\ T(v_n))]_{\mathcal{B}} = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and O are as defined in the problem statement.

Exercise 3 (Section 2.2, 15). Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V,W) : T(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S^0 is a subspace of $\mathcal{L}(V,W)$.

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

Solution 3.

(a) From theorem 1.3, a subset is a subspace iff the following properties are satisfied:

(i) $0 \in S^0$: Let $0 : V \to W,\ x \mapsto 0$ be the zero transformation. Then $0(x) = 0 \forall x \in S \subseteq V$, so $0 \in S^0$.

(ii) $T_1, T_2 \in S^0 \implies T_1 + T_2 \in S^0$: Since $T_1, T_2 \in S^0$, $T_1(x) = T_2(x) = 0$ for all $x \in S$. Then $(T_1 + T_2)(x) = T_1(x) + T_2(x) = 0 + 0 = 0$ for all $x \in S$, and so $T_1 + T_2 \in S^0$.

(iii) $cT \in S^0 \forall c \in F$: Let $T \in S^0$ and $c \in F$, so $(cT)(x) = c[T(x)] = c\dot{0} = 0$ for all $x \in S$. Thus $cT \in S^0/$

Therefore S^0 is a subspace.

(b) We prove the two-way inclusion:

$(V_1 + V_2)^0 \subseteq V_1^0 \cap V_2^0$: Let $T \in (V_1 + V_2)^0$, so $T(x) = 0$ for all $x \in V_1 + V_2$. By the definition of $V_1 + V_2$, we have

$$T(x) = T(v_1 v_2) = T(v_1) + T(v_2) \forall v_1 \in V_1, \forall v_2 \in V_2.$$

This holds for all v_1 and v_2, so $T(v_1) = T(v_2) = 0 \implies T \in V_1 \cap V_2$.

$V_1^0 \cap V_2^0 \subseteq (V_1 + V_2)^0$: Let $T \in V_1 \cap V_2$, so $T(x) = 0$ for all $x \in V_1 \cap V_2 \subseteq V_1 + V_2$. Therefore $T(x) = 0$ for all $x \in V_1 + V_2$, and $T \in (V_1 + V_2)^0$.

Thus, $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

Exercise 4 (Section 2.2, 16). Let V and W be vector spaces such that $\dim V = \dim W$, and let $T : V \to W$ be linear. Show that there exist ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively, such that $[T]_{\mathcal{B}}^{\mathcal{C}}$ is a diagonal matrix.

Solution 4. The kernel of T may be nontrivial. In this more general case, let $\{w_1, \ldots, v_m\}$ be a basis for $\ker T$, then by the replacement theorem we can extend it to a basis $\mathcal{B} = \{w_1, \ldots, w_m, v_{m+1}, \ldots, v_n\}$ for V.

We shall prove that $\{T(v_{m+1}), \ldots, T(v_n)\}$ is linearly independent. Suppose for scalars $a_{m+1}, \ldots, a_n \in F$, $a_{m+1}T(v_{m+1}) + \cdots + a_nT(a_n) = 0$, then $a_{m+1}T(v_{m+1}) + \cdots + a_nT(a_n) = 0 = T(a_{m+1}v_{m+1} + \cdots + a_nv_n)$, so $a_{m+1}v_{m+1} + \cdots + a_nv_n \in \ker T$. This means that for some scalars $a_1, \ldots, a_m \in F$,

$$a_1w_1 + \cdots + a_mw_m - (a_{m+1}v_{m+1} + \cdots + a_nv_n) = 0.$$

Since $\mathcal{B}$ is a basis, we have that $a_1 = \cdots = a_n = 0$, and the set $\{T(v_{m+1}), \ldots, T(v_n)\}$ is linearly independent. Again by the replacement theorem, we can extend it to a basis $\mathcal{C} = \{\alpha_1, \ldots, \alpha_m, T(v_{m+1}), \ldots, T(v_n)\}$ for W. In this ordered basis $\mathcal{C}$, the matrix representation of T is simply

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

with m zeroes and $n-m$ ones in the diagonal.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 5. Let V be a finite-dimensional vector space.

(a) Let U be a subspace of V with $U \neq V$. Suppose that $S \in \mathcal{L}(U, W)$ and that S is not the zero transformation. Define $T : V \to W$ by

$$T(v) = \begin{cases} S(v) & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is NOT a linear transformation on V.

(b) Prove that every linear transformation on a subspace of V can be extended to a linear transformation on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $T(u) = S(u)$ for all $u \in U$.

Solution 5.

Exercise 6. Let U, V, and W be vector spaces over a field F (not necessarily finite-dimensional).

(a) Suppose that $S : U \to V$ and $T : V \to W$ are linear transformations whose kernels are both finite-dimensional. Show that the kernel of the composition TS is also finite-dimensional and that

$$\operatorname{nullity} TS \leq \operatorname{nullity} T + \operatorname{nullity} S.$$

(b) Suppose that $S : U \to V$ and $T : V \to W$ are linear transformations whose images are both finite-dimensional. Show that the image of the composition TS is also finite-dimensional and that

$$\operatorname{rank} TS \leq \min\{\operatorname{rank} T, \operatorname{rank} S\}.$$

Solution 6.

Exercise 7. Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Suppose that $T^m = 0$ for some positive integer m. Show that $T^n = 0$, where $n = \dim V$.

Solution 7.