Homework 6

Linear Algebra I, Fall 2024
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October 19, 2024

Exercise 1 (Section 2.3, 9)**.** Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

Solution 1. Consider the transformations $U, T : F^2 \to F^2$ defined by

$$U\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix},\ T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ -(a+b) \end{pmatrix},$$

where $a, b \in F$. Here we show that U and T are indeed linear: Consider $x_1, x_2, y_1, y_2, c \in F$, then

$$U\begin{pmatrix} x_1+cy_1 \\ x_2+cy_2 \end{pmatrix} = \begin{pmatrix} x_1+cy_1+x_2+cy_2 \\ x_1+cy_1+x_2+cy_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_1+x_2 \end{pmatrix} + c\begin{pmatrix} y_1+y_2 \\ y_1+y_2 \end{pmatrix} = U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + cU\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$T\begin{pmatrix} x_1+cy_1 \\ x_2+cy_2 \end{pmatrix} = \begin{pmatrix} x_1+cy_1+x_2+cy_2 \\ -(x_1+cy_1+x_2+cy_2) \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ -(x_1+x_2) \end{pmatrix} + c\begin{pmatrix} y_1+y_2 \\ -(y_1+y_2) \end{pmatrix} = U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + cU\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We can verify that for all $a, b \in F$,

$$UT\begin{pmatrix} a \\ b \end{pmatrix} = U\begin{pmatrix} a+b \\ -(a+b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $UT = T_0$. However,

$$TU\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} 2a+2b \\ -(2a+2b) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Choose the standard ordered basis for F^2, call it β, and we have that

$$B \equiv [T]_\beta = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},\ A \equiv [U]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

satisfying $AB = O, BA \neq O$.

Exercise 2 (Section 2.3, 12)**.** Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

Solution 2.

(a) Suppose UT is one-to-one, then theorem 2.4 tells us that $\ker(UT) = \{0\}$. Now suppose there's a $v \in \ker T$ such that $T(v) = 0$, so $(UT)(v) = U(T(v)) = U(0) = 0$, and $v \in \ker(UT)$. This implies $v = 0$, and $\ker T = \{0\}$. By theorem 2.4 T is one-to-one.

In the above result, U is not necessarily one-to-one. Consider the following counterexample: let $V = Z = \mathbb{R}$ and $W = \mathbb{R}^2$. Let $T : V \to W$ and $U : W \to Z$ be given by

$$T(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}, \; U \begin{pmatrix} x \\ y \end{pmatrix} = x, \; x, y \in \mathbb{R}.$$

Then for all $x \in \mathbb{R}$ we have $UT(x) = x$, so $UT = 1$ is one-to-one and onto, but U is not one-to-one.

(b) Suppose UT is onto, then for all $z \in Z$ there exists $v \in V$ such that $(UT)(v) = z$. But then $UT(v) = U(T(v)) = z$, and we get that for all $z \in Z$, we also have $w \equiv T(v) \in W$ such that $U(w) = z$, so U is onto.

Again take the counterexample described in (a), UT is onto but T is not onto.

(c) Prove one-to-one and onto as follows:

(1) Since T and U are one-to-one, their kernels are trivial by theorem 2.4. I.e. $\ker T = \{0_v\}$, $\ker U = \{0_W\}$. Suppose $v \in \ker(UT)$, then $UT(v) = U(T(v)) = 0$, so $T(v) \in \ker U \implies T(v) = 0_W$, and then $v \in \ker T \implies v = 0_V$. So the kernel of UT is trivial, and UT is one-to-one by theorem 2.4.

(2) Suppose T and U are onto, then for all $z \in Z$ there exists $w \in W$ such that $U(w) = z$, and for all $w \in W$ there exists $v \in V$ such that $T(v) = w$. Then for all $z \in Z$ there exists $v \in V$ such that $UT(v) = U(w) = z$, so UT is onto.

Exercise 3 (Section 2.3, 13)**.** Let A and B be $n \times n$ matrices (over F). Recall that the trace of A is defined by

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$.

Solution 3. expand the matrix AB in index notation:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Then

$$\text{tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} = \text{tr}(BA),$$

where the last equal sign is because we exchange the role of dummy index i and k. For $A^\intercal$, we have $(A^\intercal)_{ij} = (A)_{ji}$, so $(A^\intercal)_{ii} = (A)_{ii}$, leaving the trace unchanged.

Exercise 4 (Section 2.4, 4)**.** Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$.

Solution 4. By assumption, the inverses A^{-1} and B^{-1} exist. Then

$$(AB)(B^{-1} A^{-1}) = I_n = (B^{-1} A^{-1})(AB),$$

so by the definition of matrix inverse, $(AB)^{-1} = B^{-1} A^{-1}$.

Exercise 5 (Section 2.4, 5)**.** Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Solution 5. First we claim that for two $n \times n$ matrices A, B, we have $(AB)^\intercal = B^\intercal A^\intercal$.

Proof.
$$[(AB)^\intercal]_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} (B^\intercal)_{ik}(A^\intercal)_{kj} = (B^\intercal A^\intercal)_{ij}.$$
□

Therefore, $(AA^{-1})^\intercal = (I_n)^\intercal = I_n = (A^{-1})^\intercal A^\intercal$, and $(A^{-1}A)^\intercal = (I_n)^\intercal = I_n = A^\intercal (A^{-1})^\intercal$. So $A^\intercal$ is invertible, with $(A^\intercal)^{-1} = (A^{-1})^\intercal$.

Exercise 6 (Section 2.4, 10)**.** Let A and B be $n \times n$ matrices such that $AB = I_n$.

(a) Prove that A and B are invertible. (If you want to use Section 2.4, 9, you need to prove it first.)

(b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are, in effect, saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)

(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

Solution 6.

(a) Here we use the result of 2.4, 9: For $n \times n$ matrices A and B, if AB is invertible, then so is A and B.

Proof. Let L_A and L_B be the left-multiplication transformation associated with matrices A and B, respectively. Then $L_A, L_B \in \mathcal{L}(V, W)$, where $\dim V = \dim W = n$. By theorem 2.18, AB is invertible iff $L_A L_B = L_{AB}$ is invertible, where equality is due to theorem 2.15. Thus $L_A L_B$ is one-to-one and onto. By exercise 2 (a) and (b) we know that L_B is one-to-one, and L_A is onto, but by theorem 2.5 one-to-one, onto, and invertible are equivalent for linear transformations in finite-dimensional spaces, so L_A and L_B are invertible. Again by theorem 2.18, A and B are invertible. □

Since $AB = I_n$ is invertible, A and B are invertible.

(b) From (a) we know that B^{-1} exists, so apply it to the right on both sides to get $ABB^{-1} = A = I_n B^{-1} = B^{-1}$.

(c) We state the following analogous result: Consider linear transformations $T, S \in \mathcal{L}(V)$, where V is finite-dimensional. Then if $TS = I_V$, T and S are invertible, with $T = S^{-1}$.

Proof. Since $TS = I_V$ is invertible, TS is one-to-one and onto, so S is one-to-one and T is onto. By theorem 2,5 T and S are both bijective, and so they are invertible. Then we verify that $TSS^{-1} = T = I_V S^{-1} = S^{-1}$. □

Definition. A relation $\sim$ on a set A is called an **equivalence relation** on A if for all $x, y, z \in A$,

(i) (reflexivity) $x \sim x$;

(ii) (symmetry) if $x \sim y$, then $y \sim x$;

(iii) (transitivity) if $x \sim y$ and $y \sim z$, then $x \sim z$.

Exercise 7 (Section 2.4, 13)**.** Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.

Solution 7. Let V, W, Z be isomorphic vector spaces, and $T \in \mathcal{L}(V, W), S \in \mathcal{L}(W, Z)$ be bijective. We check the following criteria:

(i) Let $I_V : V \to V$ be the identity map, it is linear and bijective, so it is an isomorphism, and $V \sim V$.

(ii) Suppose $V \sim W$, then there exists a bijective linear map $T : V \to W$. Consider $T^{-1} : W \to V$, by theorem 2.17 it is linear, and it is also invertible, so $W \sim V$.

(iii) Suppose $V \sim W$ and $W \sim Z$, then there exists linear bijective maps $T : V \to W$ and $S : W \to Z$.

Exercise 8 (Section 2.4, 17)**.** Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim V_0 = \dim T(V_0)$.

Solution 8.

(a) For $T(V_0)$ to be a subspace, it has to satisfy the following:

  (1) $0_W \in T(V_0)$: Since V_0 is a subspace of V, $0_V \in V_0$, then $T(0_V) = 0_W \in T(V_0)$.

  (2) $w_1 + cw_2 \in T(V_0)$ for all $w_1, w_2 \in T(V_0)$, $c \in F$: By definition of $T(V_0)$, there exists $v_1, v_2 \in V_0$ such that $w_1 = T(v_1), w_2 = T(v_2)$. Then $w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in T(V_0)$.

  By the subspace criterion, $T(V_0)$ is a subspace of W.

(b) By the rank-nullity theorem, we have

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V_0,$$

where we restrict T to the subspace V_0, so that $\operatorname{rank} T = \dim(\operatorname{im} T) = \dim T(V_0)$, and $\operatorname{nullity} T \geq 0$. In general $\dim T(V_0) \leq \dim V_0$. But since T is an isomorphism, it is also one-to-one, and by theorem 2.4 $\ker T = \{0\}$, $\dim(\ker T) = \operatorname{nullity} T = 0$, so $\dim V_0 = \dim T(V_0)$.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 9. Recall that if W is a subspace of a vector space V, we define

$$V/W := \{v + W : v \in V\}$$

to be the quotient space of V modulo W with addition and scalar multiplication defined by

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$c(v + W) = cv + W$$

for all $v \in V$ and $c \in F$.

(a) Let V and W be two vector spaces and let $T \in (V, W)$. Prove that

$$V/\ker T \simeq \operatorname{Im} T$$

with the canonical isomorphism $\overline{T} : V/\ker T \to \operatorname{Im} T$ given by

$$\overline{T}(v + \ker T) = T(v).$$

That is, you need to verify that

(i) $\overline{T}$ is well-defined; that is, prove that if $v + \ker T = v' + \ker T$, then $T(v) = T(v')$;

(ii) $\overline{T}$ is linear;

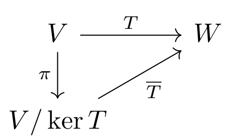
(iii) $\overline{T}$ is invertible (one-to-one and onto).

This is called the **first isomorphism theorem**.

(b) Let $\pi : V \to V/\ker T$ be the natural **quotient map** defined by

$$\pi(v) = v + \ker T,$$

which is clearly a linear transformation. Prove that the following diagram commutes;



that is, prove that $T = \overline{T} \circ \pi$. In this case, we say that T **factors through** $V/\ker T$.

(c) Assume that V is finite-dimensional. Use (a) to deduce the rank-nullity theorem.

(d) Let U and W be subspaces of a vector space V. Prove that

$$(U + W)/W \simeq U/(U \cap W)$$

by (a) via the map $T : U \to (U + W)/W$ defined by

$$T(u) = u + W.$$

This is called the **second isomorphism theorem**.

(e) Let U and W be finite-dimensional subspaces of a vector space V. Use (d) to deduce that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Solution 9.

(a) We show the following:

(i) $\overline{T}$ is well-defined: Take $v, v' \in V$ such that $v + \ker T = v' + \ker T$, by problem 1.3,31 $v - v' \in \ker T$, so there exists some $k \in \ker T$ such that $v - v' = k$. Then $T(v) = T(v'+k) = T(v') + T(k) = T(v')$. Therefore, $v + \ker T = v' + \ker T \implies \overline{T}(v + \ker T) = \overline{T}(v' + \ker T)$.

(ii) $\overline{T}$ is linear: Let $v_1 + \ker T, v_2 + \ker T \in V/\ker T$ and $c \in F$, then $\overline{T}((v_1 + \ker T) + c(v_2 + \ker T)) = \overline{T}((v_1 + cv_2) + \ker T) = T(v_1 + cv_2)$. But T is linear, so this equals $T(v_1) + cT(v_2) = \overline{T}(v_1 + \ker T) + c\overline{T}(v_2 + \ker T)$.

(iii) $\overline{T}$ is invertible: First let $v + \ker T \in \ker \overline{T}$, then $\overline{T}(v + \ker T) = T(v) = 0$, so $v \in \ker T \implies v + \ker T = 0 + \ker T$, and $\ker \overline{T} = \{0 + \ker T\}$, so by theorem $\overline{T}$ is one-to-one. Next take some $u \in \operatorname{im} T$, so there exists $v \in V$ such that $T(v) = u$. Then for all $u \in \operatorname{im} T$, $\overline{T}(v + \ker T) = T(v) = u$, so $\overline{T}$ is onto. Thus $\overline{T}$ is invertible.

From the above discussion, $V/\ker T \simeq \operatorname{im} T$.

(b) Take some arbitrary $v \in V$, then $(\overline{T} \circ \pi)(v) = \overline{T}(v + \ker T) = T(v)$. This holds for all $v \in V$, so $\overline{T} \circ \pi = T$, and the diagram commutes.

(c) Assume $\dim V < \infty$, let β be a finite basis for V. By theorem 2.2 we have $\operatorname{im} T = \operatorname{span}(T(\beta))$, and by theorem there exists a linearly indepedent subset of $T(\beta)$ that is a basis for $\operatorname{im} T$, so $\dim \operatorname{im} T < \infty$.

Consider the map $\overline{T} : V/\ker T \to \operatorname{im} T$ as defined above, by the first isomorphism theorem $V/\ker T \simeq \operatorname{im} T$. So by theorem 2.19 they have the same dimension: $\operatorname{rank} T = \dim \operatorname{im} T = \dim(V/\ker T) = \dim V - \dim \ker T = \dim V - \operatorname{nullity} T$, where the second last inequality is by problem 2.1,40.

(d) Suppose $u \in \ker T$, then $T(u) = u + W = 0 + W$, so $u \in W \implies \ker T \subseteq U \cap W$. Conversely, take $u \in U \cap W$, then $T(u) = u + W = 0 + W$. Therefore $\ker T = U \cap W$. Next, for all $u + W \in (U + W)/W$, $T(u) = u + W$, so T is onto and $\operatorname{im} T = (U + W)/W$. By the first isomorphism theorem, $U/(U \cap W) \simeq (U + W)/W$.

(e) Again use problem 2.1,40 and theorem 2.19,

$$\begin{aligned} \dim((U + W)/W) &= \dim(U + W) - \dim W = \dim(U/(U \cap W)) \\ &= \dim U - \dim(U \cap W). \end{aligned}$$

Then $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.