Homework 7

Linear Algebra I, Fall 2024
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October 23, 2024

Definition. Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. A function $T : V \to V$ is called the **projection on** W_1 **along** W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$.

Exercise 1 (Section 2.5, 7)**.** In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L.

Solution 1.

(a) From geometry, the reflection matrix is

$$R(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix},$$

where $\tan\theta = m$. So we have

$$T(x, y) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{m^2 + 1}\left((1 - m^2)x + 2my, 2mx - (1 - m^2)y\right).$$

The perpendicular unit vector is

$$\frac{1}{\sqrt{1 + m^2}(-m, 1)},$$

and the magnitude of the projected vector is $\sqrt{x^2 + y^2}\sin\phi$, where ϕ is the angle between (x, y) and L. So

$$T(x, y) = \sqrt{x * 2 + y^2}\sin((\phi + \theta) - \theta)(-m, 1)/\sqrt{1 + m^2},$$

plug in the formulas to get

$$\sqrt{x^2 + y^2}\left[\frac{y}{\sqrt{x^2 + y^2}\frac{1}{\sqrt{1+m^2}} - \frac{m}{\sqrt{1+m^2}}\frac{x}{\sqrt{x^2 y^2}}}\right] = \frac{1}{\sqrt{1 + m^2}}(y - mx).$$

So we have

$$T(x, y) = \frac{1}{1 + m^2}(y - mx)(-m, 1).$$

Exercise 2 (Section 2.5, 9)**.** Prove that "is similar to" is an equivalence relation on $M_{n\times n}(F)$.

Solution 2. Denote matrix similarity by the symbol $\sim$, we will show that $\sim$ satisfies the rquirements of an equiva;ence equation:

1. Reflexivity: For an $n \times n$ matrix A, I_n is invertible, so $A = I_n^{-1} A I_n$ implies $A \sim A$.
2. Symmetry: Suppose two $n \times n$ matrices A, B satisfy $A \sim B$, so there exists Q such that $B = Q^{-1}AQ$. Then $A = QBQ^{-1} = (Q^{-1})^{-1}BA^{-1}$, so $B \sim A$.

3. Transitivity: Suppose $A \sim B$ and $B \sim C$, then there exists invertible $n \times n$ matrices P, Q such that $B = Q^{-1}AQ$ and $C = P^{-1}BP$. Then $C = P^{-1}Q^{-1}AQP = (QP)^{-1}A(QP)$, so $A \sim C$.

Exercise 3 (Section 2.5, 10)**.** Prove that if A and B are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$. *Hint*: Use Section 2.3, 13.

Solution 3. Assume $A \sim B$, using the notation from last problem. Then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Recall the property of the trace that for three arbitrary matrices A, B, C (such that the product is defined), $\text{tr}(ABC) = \text{tr}(CAB)$.

Thus, $\text{tr}(B) = \text{tr}(Q^{-1}AQ) = \text{tr}(QQ^{-1}A) = \text{tr}(A)$.

Exercise 4 (Section 2.5, 13)**.** Let V be a finite-dimensional vector space over a field F, and let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \le j \le n,$$

and set $\mathcal{B}' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that $\mathcal{B}'$ is a basis for V and hence that Q is the change of coordinate matrix changing $\mathcal{B}'$-coordinates into $\mathcal{B}$-coordinates.

Solution 4. We first show that $\mathcal{B}'$ is a spanning set:

$$\begin{aligned}\text{span}(\mathcal{B}') = \text{span}\left(\{x'_1, \ldots, x'_n\}\right) &= \sum_{j=1}^{n} a_j x'_j \\ &= \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{n} Q_{ij} x_i\right) = \sum_{j=1}^{n} \left(a_j \sum_{i=1}^{n} Q_{ij}\right) x_i \\ &= \text{span}\left(\{x_1, \ldots x_n\}\right) = \text{span}(\mathcal{B}) = V.\end{aligned}$$

By corollary 2 (a) of theorem 1.10, a spanning set with size $\dim V$ is a basis, so $\mathcal{B}'$ is a basis for V. Thus, Q is the change of coordinate matrix from $\mathcal{B}'$ to $\mathcal{B}$. .

Exercise 5 (Section 3.2, 14)**.** Let $T, U : V \to W$ be linear transformations.

(a) Prove that $\text{Im}(T + U) \subseteq \text{Im}\, T + \text{Im}\, U$.

(b) Prove that if W is finite-dimensional, then $\text{rank}(T + U) \le \text{rank}\, T + \text{rank}\, U$.

(c) Deduce from (b) that $\text{rank}(A + B) \le \text{rank}\, A + \text{rank}\, B$ for any $m \times n$ matrices A and B.

Solution 5.

(a) Take some arbitrary $v \in V$, then by definition $(T + U)(v) \in \text{im}(T + U)$. Then $(T + U)(v) = T(v) + U(v) \in \text{im}(T) + \text{im}(U)$, so $\text{im}(T + U) \subseteq \text{im}(T) + \text{im}(U)$.

(b) From HW3 ex.7, we have the result that when W is finite dimensional, then two subspaces $W_1, W_2 \subseteq W$ satisfy

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Notice that $\text{im}\, T$ and $\text{im}\, U$ are subspaces of W, so we have

$$\begin{aligned}\dim(\text{im}(T + U)) &\le \dim(\text{im}\, T + \text{im}\, U) \\ &= \dim(\text{im}\, T) + \dim(\text{im}\, U) - \dim(\text{im}\, T \cap \text{im}\, U) \\ &\le \dim(\text{im}\, T) + \dim(\text{im}\, U).\end{aligned}$$

Here the first inequality is due to (a), and the second is due to the above mentioned result.

(c) Let $V = F^n$, $W = R^m$, $T = L_A$, and $U = L_B$ in (b). From theorem 2.15 (iii) we have that $L_{A+B} = L_A + L_B$, so $\operatorname{rank}(L_{A+B}) = \operatorname{rank}(L_A + L_B) \leq \operatorname{rank} L_A + \operatorname{rank} L_B$. Thus, $\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$, as desired.

Exercise 6 (Section 3.2, 20)**.** Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}.$$

(a) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

(b) Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $\operatorname{rank} B \leq 2$.

Solution 6. (a) Let the matrix M be given by

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 \\ a_3 & b_3 & 0 & 0 & 0 \\ a_4 & b_4 & 0 & 0 & 0 \\ a_5 & b_5 & 0 & 0 & 0 \end{pmatrix},$$

which is rank 2 by theorem 3.5 as required by the problem statement. Solving for the tuple $(a_1, a_2, a_3, a_4, a_5)$, and similarly for $(b_1, \ldots, b_5)$, gives infinitely many solutions, we choose the following solution

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & -7 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

which has rank 2 by theorem 3.5.

(b) By theorem in textbook, any elementary operation on the matrix would preserve its rank, so the rank cannot be any larger than the example given in (a), that is, 2.

Exercise 7 (Section 3.2, 21)**.** Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

Solution 7.

Claim. A map $T : V \to W$ has a right inverse if and inly if T is onto.

Proof. We prove by the two directions: ($\implies$) Suppose there exists $S : W \to V$ such that $TS : W \to W$ is the identity map. Since the identity map is onto, T is also onto by HW6 ex. 2.

($\impliedby$) Suppose now T is onto, then for every $w \in W$ there is some $v \in V$ such that $T(v) = w$. Therefore we can define the preimage of T, which we call S, such that $S : W \to V$ is given by $S(w) = v$ for all $w \in W$, where v is chosen such that $T(v) = w$. Then for all $w \in W$, we have $(TS)(w) = T(v) = w$, so $TS = I_W$. □

Using the above result, we have that A has a right inverse iff $L_A : F^n \to F^m$ has a right inverse, iff L_A is onto, iff $\operatorname{im} L_A = F^m$, iff $\dim(\operatorname{im} L_A) = \dim F^m = m$. The last iff is true because the dimension of F^m and $\operatorname{im} L_a$ is finite dimensional, and $\operatorname{im} L_A \subseteq F^m$, so by theorem in textbook they are equal. Therefore, A has a right inverse iff $\operatorname{rank} A = \operatorname{rank} L_A = m$.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 8. Define

$$J = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \in M_{n\times n}(\mathbb{R})$$

to be the matrix whose entries are all equal to 1. Consider the following subspace of $M_{n\times n}(\mathbb{R})$

$$W = \{A \in M_{n\times n}(\mathbb{R}) : AJ = JA\}.$$

Find a basis and the dimension of W in terms of n.

Exercise 9. Let $A \in M_{2\times 2}(\mathbb{R})$ be a nonzero matrix such that $\operatorname{tr}(A) = 0$.

(a) Show that there exists a vector $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is a basis for $\mathbb{R}^2$.

(b) Show that there exists an invertible matrix $P \in M_{2\times 2}(\mathbb{R})$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

for some $b, c \in \mathbb{R}$.

Exercise 10. Let $A \in M_{n\times n}(F)$. Define L_A and R_A to be the linear transformations from $M_{n\times n}(F)$ to $M_{n\times n}(F)$ by

$$L_A(B) = AB, \quad R_A(B) = BA.$$

Show that $\operatorname{rank} L_A = \operatorname{rank} R_A$.