Homework 8

Linear Algebra I, Fall 2024

黃紹凱 B12202004

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Exercise 1 (Section 3.3, 6)**.** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(a, b, c) = (a + b, 2a - c).$$

Determine $T^{-1}(1, 11)$. (Here, T^{-1} means the preimage of T).

Solution 1. Recall the definition of the set-theoretic preimage: for $x, y \in \mathbb{R}$,

$$T^{-1}(x, y) \equiv \left\{(a, b, c) \in \mathbb{R}^3 \,\middle|\, T(a, b, c) = (x, y)\right\}.$$

Suppose $T(a, b, c) = (1, 11)$, then

$$\begin{cases} a + b &= 1, \\ 2a - c &= 11. \end{cases} \implies \begin{cases} b &= 1 - a, \\ c &= 2a - 11. \end{cases}$$

So

$$T^{-1}(1, 11) = \{(a, 1 - a, 2a - 11) \mid a \in \mathbb{R}\}.$$

Exercise 2 (Section 3.3, 9)**.** Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in \operatorname{Im}(L_A)$.

Solution 2.

($\implies$): Suppose $s \in F^n$ is a solution to $Ax = b$, then $As = L_A(s) = b$, so $b \in \operatorname{Im} L_A$.

($\impliedby$): Suppose $b \in \operatorname{Im} L_A$, then there exists some $s \in F^n$ such that $L_A(s) = b$. Then $L_A(s) = As = b$, so s is a solution to $Ax = b$.

Exercise 3 (Section 3.3, 10)**.** Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

Solution 3. $\operatorname{rank} A = \operatorname{rank} L_A = \dim \operatorname{Im} L_A = m = \dim F^m$. Since $\operatorname{Im} L_A$ and F^m are finite-dimensional, by theorem in textbook we have $\operatorname{Im} L_A = F^m$. Thus L_A is surjective, so for any $b \in F^m$, there exists some solution $s \in F^m$ such that $L_A(s) = As = b$.

Exercise 4 (Section 3.4, 3)**.** Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

(a) Prove that $\operatorname{rank} A' \neq \operatorname{rank}(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

(b) Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

Solution 4.

(a) We prove both directions:

($\Longrightarrow$): Suppose $\operatorname{rank}(A') \neq \operatorname{rank}(A'|b')$. Then it must be the case that $(A'|b')$ has one more linearly independent column (the $(m+1)$-th column) of the form e_i $(r+1 \leq i \leq n)$, where $r = \operatorname{rank} A$, since if $1 \leq i \leq r$, by theorem 3,16 (b) there is a column $b_{j(i)} = e_i$, so the new column would not be linearly independent from the rest.

By definition of the reduced row echelon form, the i-th row $(r+1 \leq i \leq n)$ of A' is all zero, so it is a row where the only nonzero entry lies in the last column.

($\Longleftarrow$): Suppose $(A'|b')$ contains a row in which the only nonzero entry lies in the last column. Suppose this row is the j-th row $(r+1 \leq j \leq n)$. Suppose a solution $x \in F^n$ exists, then this row corresponds to the linear equation of n unknowns

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn} = 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 0 = c \neq 0.$$

Contradiction, so there cannot be a solution, $Ax = b$ is inconsistent. By theorem 3.11 $\operatorname{rank}(A'|b') \neq \operatorname{rank}(A')$.

(b) Take the contrapositive on both sides of the result of (a), then apply theorem 3.11, which says the fact that a system $Ax = b$ is consistent is equivalent to the equality $\operatorname{rank} A' = \operatorname{rank}(A'|b')$.

Exercise 5 (Section 3.4, 7)**.** It can be shown that the vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

Solution 5. Let

$$A = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix}.$$

We can use Gaussian elimination to put A into reduced row echelon form:

$$A = \begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 5 & 0 & 35 & -19 \\ 0 & -2 & 0 & -14 & 19 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 3 & 0 & 21 & 0 \\ 0 & 5 & 0 & 35 & 19 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & - & -4 & -3 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \equiv B.$$

B is in reduced row echelon form. Let b_i denote the i-th column of B, we have $b_1 = e_1$, $b_2 = e_2$, and $b_5 = e_3$. By theorem 3.16 (c) a_1, a_2, a_5 are linearly independent. Since $\dim \mathbb{R}^3 = 3$, by corollary to theorem 1.10 $\{a_1, a_2, a_5\}$ is a basis. Therefore the desired subset is

$$\{u_1, u_2, u_5\} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 8 \end{pmatrix} \right\}.$$

Exercise 6 (Section 3.4, 12)**.** Let V denote the set of all solutions to the system of linear equations

$$\begin{aligned} x_1 - x_2 \quad\quad + 2x_4 - 3x_5 + \ x_6 &= 0 \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 &= 0. \end{aligned}$$

(a) Show that $S = \{(0, -1, 0, 1, 1, 0),\ (1, 0, 1, 1, 1, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

Solution 6.

(a) By substituting $(0,-1,0,1,1,0)$ and $(1,0,1,1,1,0)$ into the system of equations, we see thet are solutions, so S is indeed a subset of V. Now we check for linear independence: Suppose there exists scalrs c_1, c_2 such that

$$c_1 \begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}^\top + c_2 \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}^\top = 0$$

Then

$$\begin{pmatrix} c_2 & -c_1 & c_2 & c_1 + c_2 & c_1 + c_2 & 0 \end{pmatrix}^\top = 0,$$

so $c_1 = c_2 = 0$, and S is a linearly independent subset of V.

(b) We write the equation as $Ax = 0$:

$$Ax = \begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

V is the solution set to the system $Ax = 0$. $L_A : F^6 \to F^2$ is the left-multiplication transformation associated with A. Notice the reduced row echelon form of A

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix} \equiv B.$$

$\operatorname{rank} A = \operatorname{rank} B = 2$, so by the rank-nullity theorem we have $\dim V = \dim \ker L_A = n - \operatorname{rank} L_A = 6 - 2 = 4$. We need to find 4 linearly independent vectors in V. Consider the augmented matrix $(A|0) \longrightarrow (B|0)$, following the procedure outlined in the textbook, let $t_1 = x_3$, $t_2 = x_4$, $t_3 = x_5$, and $t_4 = x_6$. We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Collect the 4 vectors above into $\mathcal{B}$, then $\mathcal{B} \cup S$ is a spanning set of V. Notice that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

so we remove these two vectors, leaving us with

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

as a basis of V containing S.

Exercise 7 (Section 3.4, 15)**.** Prove that the reduced row echelon form of a matrix is unique using Theorem 3.16.

Solution 7. From theorem 3.16, let $\operatorname{rank} A = r$, then every column of the rediced row echelon form of A (which we will call B) is of the form

$$d_1 e_1 + \cdots + d_r e_r,$$

for scalars $d_1, \ldots, d_r$. Let the k-th column of B be denoted b_k, then (d) of theorem 3.16 says that if

$$b_k = d_{1,k} e_1 + \cdots + d_{r,k} e_r \implies a_k = d_{1,k} a_{j(1)} + \cdots + d_{r,k} a_{j(r)},$$

where $a_{j(1)}, \ldots, a_{j(r)}$ are linearly independent by (c) of the same theorem.

We claim that given a_k ($1 \leq k \leq n$) all fixed, $d_{1,k}, \ldots, d_{r,k}$ are uniquely determined. Suppose there are scalars $d_{1,k}, \ldots, d_{r,k}, d'_{1,k}, \ldots, d'_{r,k}$ such that

$$a_k = d_{1,k} a_{j(1)} + \cdots + d_{r,k} a_{j(r)} = d'_{1,k} a_{j(1)} + \cdots + d'_{r,k} a_{j(r)}.$$

Then

$$a_k - a_k = 0 = (d_{1,k} - d'_{1,k}) e_1 + \cdots + (d_{r,k} - d'_{r,k}) e_r.$$

By the linear independence of $a_{j(k)}$, we have $d_{1,k} = d'_{1,k}$, ..., $d_{r,k} = d'_{r,k}$. Therefore

$$b_k = d_{1,k} e_1 + \cdots d_{r,k} e_r$$

is unique.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 8.

(a) Show that for any $A \in M_{n\times n}(\mathbb{R})$, there exists $B \in M_{n\times n}(\mathbb{R})$ such that

$$AB = O \quad \text{and} \quad \operatorname{rank} A + \operatorname{rank} B = n.$$

(b) Prove or give a counterexample that for any $A \in M_{n\times n}(\mathbb{R})$, there exists $B \in M_{n\times n}(\mathbb{R})$ such that

$$AB = BA = O \quad \text{and} \quad \operatorname{rank} A + \operatorname{rank} B = n.$$

Solution 8.

(a) Notice that the condition $\operatorname{rank} A + \operatorname{rank} B = n$ hints at a use of the rank-nullity theorem. Suppose $\dim \ker L_A = k$, then let $\mathcal{B} = \{v_1, \dots, v_k\}$ be a basis for $\ker L_A$. Then we have

$$L_A(v_1) = \cdots = L_A(v_k) = 0 \Leftrightarrow Av_1 = \cdots Av_k = 0,$$

where in the second set of equalities $v_1, \dots, v_k$ are the coordinate vectors of $v_1, \dots, v_k$ in the canonical basis pf $\mathbb{R}^n$. Let

$$B = \begin{pmatrix} v_1 & \cdots & v_k & 0 & \cdots & 0 \end{pmatrix} \implies AB = \begin{pmatrix} Av_1 & \cdots & Av_k & 0 & \cdots & 0 \end{pmatrix} = O.$$

Furthermore, since $\operatorname{rank} B = |\mathcal{B}| = k = \operatorname{nullity} A$, $\operatorname{rank} A + \operatorname{rank} B = n$ by the rank-nullity theorem.

(b) To be finished...

Exercise 9. Let $A \in M_{n\times n}(\mathbb{C})$ be strictly column diagonally dominant, i.e.,

$$|A_{ii}| > \sum_{1\le j\le n,\ j\neq i} |A_{ji}|, \quad i = 1, \dots, n.$$

Show that A is invertible.
Hint: Show that the system of linear equations $Ax = 0$ has only trivial solution.

Solution 9. Assume there is some vector $0 \neq x \in \mathbb{C}^n$ such that $Ax = 0$. Then there exists some i $(1 \le i \le n)$ such that $|x_i| \ge |x_k|$ for all $1 \le k \le n$. Then $x_i \neq 0$, and

$$\sum_{j=1}^{n} a_{ij}x_j = 0$$

implies

$$a_{ii}x_i = \sum_{j\neq i} a_{ij}x_j.$$

Then

$$\begin{aligned} |a_{ii}vert &= \left|\sum_{j\neq i} a_{ij}\frac{x_j}{x_i}\right| \\ &\le \sum_{j\neq i}\left|a_{ij}\frac{x_j}{x_i}\right| \\ &= \sum_{j\neq i}|a_{ij}|\left|\frac{x_j}{x_i}\right| \le \sum_{j\neq i}|a_{ij},| \end{aligned}$$

contradiction. So $x = 0$ is the only solution of $Ax = 0$. $\ker L_A = \{0\}$ implies L_A is injective, and thus invertible, which then implies A is also invertible.

Definition. A **graph** $G = (V, E)$ is a pair of finite sets V, E, and any element in E is a pair of elements $\{u, v\}$ in V. V is called the **vertex set** and E is called the **edge set**.

(This definition may appear frequently in the later extra exercises.)

A **walk** of length ℓ from u to w is a sequence $u = v_0, v_1, v_2, \ldots, v_\ell = w$ of vertices for which $\{v_{i-1}, v_i\} \in E$ for $i = 1, 2, \ldots, \ell$.

Let $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$. The **adjacency matrix** A of the graph G, denoted by A_G, is the $n \times n$ matrix defined by

$$(A_G)_{ij} = \begin{cases} 1, & \{v_i, v_j\} \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 10.

(a) Show that for any bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, the adjacency matrix of G defined by $\{v_1, \ldots, v_n\}$ and $\{v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}\}$ are similar.

(b) What is $\operatorname{tr}({A_G}^2)$?

(c) Fix a order of vertices $\{v_1, \ldots, v_n\}$ and a positive integer k. Show that $({A_G}^k)_{ij}$ is the number of walks of length k, from v_i to v_j.

Solution 10.