Homework 9

Linear Algebra I, Fall 2024

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Exercise 1 (Section 4.1, 9). Prove that $\det(AB) = (\det A)(\det B)$ for any $A, B \in M_{2\times 2}(F)$.

Exercise 2 (Section 4.1, 11). Let $\delta : M_{2\times 2}(F) \to F$ be a function with the following three properties.

(i) δ is a linear function of each row of the matrix when the other row is held fixed.

(ii) If the two rows of $A \in M_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.

(iii) If I_2 is the 2×2 identity matrix, then $\delta(I_2) = 1$.

Prove that $\delta(A) = \det A$ for all $A \in M_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

Exercise 3 (Section 4.2, 4). Find the value of k that satisfies the following equation:

$$\det \begin{pmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ a_1 + c_1 & a_2 + c_2 & a_3 + c_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{pmatrix} = k \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Exercise 4 (Section 4.2, 12). Evaluate the determinant of

$$\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$$

by cofactor expansion along the fourth row.

Exercise 5 (Section 4.2, 22). Evaluate the determinant of

$$\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}.$$

Exercise 6 (Section 4.2, 23). Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Exercise 7 (Section 4.2, 25). Prove that $\det kA = k^n \det A$ for any $A \in M_{n\times n}(F)$ and $k \in F$.

Exercise 8 (Section 4.2, 26). Let $A \in M_{n\times n}(F)$. Under what conditions is $\det(-A) = \det A$?

Exercise 9 (Section 4.2, 30). Let the rows of $A \in M_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let $B \in M_{n\times n}(F)$ be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det B$ in terms of $\det A$.

(There are extra exercises in the next page.)

Extra Exercises

You don't have to hand in extra exercises, and solving them will NOT affect your grade.

Exercise 10. Let $A \in M_{n\times n}(F)$ such that the sums of each column and each row are 0. That is,

$$\sum_{i=1}^{n} A_{ij} = \sum_{i=1}^{n} A_{ji} = 0 \quad \text{for all } 1 \leq j \leq n.$$

(a) Show that $\det A = 0$.

(b) Show that the cofactors of A are all identical. That is, show that $(-1)^{i+j} \det \tilde{A}_{ij}$ are the same for every $1 \leq i, j \leq n$.

Exercise 11. Consider the $n \times n$ matrix

$$F_n = \left(\binom{i-1+j-1}{i-1} \right)_{1 \leq i,j \leq n} \in M_{n\times n}(\mathbb{N}).$$

For example,

$$F_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}, \ldots$$

Show that $\det F_n = 1$ for all $n \in \mathbb{N}$.