2025 Fall Introduction to ODE

Homework 1 (Due Sep 8, 2025)

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Problem 1. Find the two linearly independent solutions of each of the differential equations

(a) $x^2 \frac{d^2y}{dx^2} + x\left(x - \frac{1}{2}\right) \frac{dy}{dx} + \frac{1}{2}y = 0,$

(b) $x^2 \frac{d^2y}{dx^2} + x(x+1)\frac{dy}{dx} - y = 0,$

using the method of Frobenius

Solution 1. Given an ordinary differential equation of the form

$$x^2 \frac{d^2y}{dx^2} + x\,p(x)\frac{dy}{dx} + q(x)y = 0, \tag{1}$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$, the existence of two linearly independent solutions as a Frobenius series

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

with $a_0 \neq 0$ around the regular singular point $x = 0$ is guaranteed by Fuchs Theorem. It is obvious that $x = 0$ is not a solution to (a) nor (b), and the corresponding $p(x)$ and $q(x)$ are analytic at $x = 0$. Thus, we can apply the method of Frobenius.

(a) Assume the solution is of the form $y(x) = x^r \sum_{n=1}^{\infty} a_n x^n$.

(b) Assume the solution is of the form $y(x) = x^r \sum_{n=1}^{\infty} a_n x^n$.