2025 Fall Introduction to ODE

Homework 10 (Due November 24 12:00, 2025)

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Exercise 1. Let $f(y,t)$ and $g(y,t)$ be continuous and satisfy a Lipschitz condition with respect to y in a region D. Suppose $|f(y,t)-g(y,t)| < \varepsilon$ in D for some $\varepsilon > 0$. If $y_1(t)$ is a solution of $y' = f(y,t)$ and $y_2(t)$ is a solution of $y' = g(y,t)$, such that $|y_2(t_0) - y_1(t_0)| < \delta$ for some t_0 and $delta > 0$. Show that for all t for which $y_1(t)$ and $y_2(t)$ both exist, we have the inequality

$$|y_2(t) - y_1(t)| \leq \delta \exp(K|t-t_0|) + \frac{\varepsilon}{K}\{\exp(K|t-t_0|) - 1\},$$

where K is the Lipschitz constant. Hint: Use the Gronwall inequality.

Solution 1.

Steps:

1. Bound $|y_2'(t) - y_1'(t)|$ using the Lipschitz condition and the given inequality.
2. Apply Gronwall's inequality to obtain the desired bound.

Method:

1. Let K be the Lipschitz constant of both f and g with respect to y. Consider the difference

$$\begin{aligned}|y_2'(t) - y_1'(t)| &= |f(y_1(t),t) - g(y_2(t),t)| \\ &\leq |f(y_1(t),t) - f(y_2(t),t)| + |f(y_2(t),t) - g(y_2(t),t)| \\ &\leq K\,|y_1(t) - y_2(t)| + \varepsilon.\end{aligned}$$

Let $u(t) = |y_2(t) - y_1(t)|$, then we have

$$u'(t) = \frac{y_2(t) - y_1(t)}{|y_2(t) - y_1(t)|}\left(y_2'(t) - y_1'(t)\right) \leq |y_2'(t) - y_1'(t)| \leq Ku(t) + \varepsilon.$$

2. By assumption, we have $u(t_0) = |y_2(t_0) - y_1(t_0)| < \delta$. Let $v(t) = u(t) + \frac{\varepsilon}{K}$, then

$$v'(t) \leq Kv(t), \quad v(t_0) < \delta + \frac{\varepsilon}{K}.$$

By Gronwall's inequality, we have

$$v(t) \leq v(t_0)\exp(K|t-t_0|) < \left(\delta + \frac{\varepsilon}{K}\right)\exp(K|t-t_0|).$$

Since $u(t) = |y_2(t) - y_1(t)|$, for all $t \geq t_0$, we have

$$\begin{aligned}u(t) = v(t) - \frac{\varepsilon}{K} &< \left(\delta + \frac{\varepsilon}{K}\right)\exp(K|t-t_0|) - \frac{\varepsilon}{K} \\ &= \delta\exp(K|t-t_0|) + \frac{\varepsilon}{K}\left(\exp(K|t-t_0|) - 1\right).\end{aligned}$$

Similarly, for all $t < t_0$, apply the same reasoning to the interval $[t, t_0]$ gives

$$\begin{aligned}u(t) = v(t) - \frac{\varepsilon}{K} &< \left(\delta + \frac{\varepsilon}{K}\right)\exp(K|t-t_0|) - \frac{\varepsilon}{K} \\ &= \delta\exp(K|t-t_0|) + \frac{\varepsilon}{K}\left(\exp(K|t-t_0|) - 1\right).\end{aligned}$$

Exercise 2. Let $\sigma(t) \in C^1([a, a+\varepsilon])$, $\sigma(t) > 0$, and $0 < \sigma(a) \leq 1$. Suppose $\sigma(t)$ satisfies the differential inequality $\sigma' \leq K\sigma \log \sigma$, show the inequality

$$\sigma(t) \leq \sigma(a)e^{K(t-a)}, \quad \text{for } t \in [a, a+\varepsilon].$$

Solution 2.

Steps:

1. Consider a change of variables to simplify the differential inequality.
2. Apply Gronwall's inequality to derive the desired bound.

Method:

1. Consider the function $\phi(t) = \log \sigma(t)$. Since $\sigma(t) > 0$, $\phi(t)$ is well-defined and differentiable on $[a, a+\varepsilon]$. We have $\phi(a) = \log \sigma(a) < 0$ and

$$\sigma(t)\phi'(t) = \sigma'(t) \leq K\sigma(t) \log \sigma(t).$$

Since $\sigma(t) > 0$, divide both sides by $\sigma(t)$ to obtain

$$\phi'(t) \leq K\phi(t).$$

2. Apply Gronwall's inequality to $\phi(t)$ on the interval $[a, t]$ for any $t \in [a, a+\varepsilon]$, we have

$$\phi(t) \leq \phi(a)e^{K(t-a)} = \log \sigma(a)e^{K(t-a)}.$$

Exponentiating both sides now gives

$$\sigma(t) = e^{\phi(t)} \leq e^{\log \sigma(a)e^{K(t-a)}} = \sigma(a)^{e^{K(t-a)}}.$$

To obtain the desired inequality, note that since $0 < \sigma(a) < 1$, we have $\phi(a) < 0$ and $K(t-a) > 0$ for $t \in [a, a+\varepsilon]$. Then

$$\phi(t) \leq \phi(a)e^{K(t-a)} \leq \phi(a)\left(1 + K(t-a)\right) \leq \phi(a) + K(t-a),$$

which implies by exponentiation that

$$\sigma(t) = e^{\phi(t)} \leq e^{\phi(a)+K(t-a)} = \sigma(a)e^{K(t-a)}, \quad t \in [a, a+\varepsilon].$$

Exercise 3. For each fixed x, let $F(x,y)$ be a nonincreasing function of y. Show that, if $f(x)$ and $g(x)$ are two solutions of $y' = F(x,y)$ and $b > a$, then

$$|f(b) - g(b)| \leq |f(a) - g(a)|.$$

Hence, deduce a result concerning the uniqueness of solutions. This is known as the **Peano uniqueness theorem**.

Solution 3.

Steps:

1. Show that $u(x) = |f(x) - g(x)|$ is a nonincreasing function of x.
2. Deduce the Peano uniqueness theorem as a consequence.

Method:

1. Consider the difference $u(x) = |f(x) - g(x)|$. Then,

$$u'(x) = \frac{f(x) - g(x)}{u(x)} \left(f'(x) - g'(x)\right) = \frac{f(x) - g(x)}{u(x)} \left(F(x, f(x)) - F(x, g(x))\right).$$

Suppose $f(x) > g(x)$, then $u(x) = f(x) - g(x) > 0$. However, since $F(x,y)$ is nonincreasing in y, we have $F(x, f(x)) - F(x, g(x)) \leq 0$, and hence $u'(x) \leq 0$. On the other hand, if $f(x) < g(x)$, then $u(x) > 0$ and $u'(x) \leq 0$ again. Therefore, we have

$$u(x)u'(x) \leq 0 \implies \frac{d}{dx}(u(x))^2 = 2u(x)u'(x) \leq 0,$$

while the case $f(x) = g(x)$ is trivial. Therefore, $(u(x))^2$ is a nonincreasing function for all $x \in \mathbb{R}$. Since $u(x) \geq 0$, $u(x)$ is also nonincreasing on $\mathbb{R}$, and we have

$$|f(b) - g(b)| = |u(b)| \leq |u(a)| = |f(a) - g(a)|, \quad \text{for all } b > a.$$

2. As a direct consequence, suppose f and g are two solutions to the initial value problem $y'(x) = F(x, y(x))$, subject to the same initial condition $f(x_0) = g(x_0)$ for some $x_0 \in \mathbb{R}$, and $F(x,y)$ is a nondecreasing function in y. Then $f(x) = g(x)$ for all $x > x_0$. Thus, the solution to the initial value problem $y' = F(x,y)$, $y(x_0) = y_0$ is unique.

This result may be what is referred to as the Peano uniqueness theorem in the problem statement.