2025 Fall Introduction to ODE

Homework 11 (Due December 1 12:00, 2025)

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Exercise 1. Consider the second-order, autonomous ordinary differential equation

$$\ddot{x} = 3x^2 - 1,$$

where a dot represents $\frac{d}{dt}$. By integrating this equation once, obtain a relation between $\dot{x}$ and x. Sketch the phase portrait in the $(x, \dot{x})$-phase plane. Determine the coordinates of the two equilibrium points.

Solution 1.

Steps:

1. Integrate once to obtain a relation between $\dot{x}$ and x by multiplying both sides by $\dot{x}$.
2. Sketch the phase portrait in the $(x, \dot{x})$-phase plane.

Method:

1. Multiply both sides by $\dot{x}$, and notice that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\ddot{x}$ by the chain rule:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\dot{x}^2 = \ddot{x}\dot{x} = (3x^2 - 1)\dot{x}.$$

Then, integrate both sides with respect to t:

$$\frac{1}{2}\dot{x}^2 = \int \mathrm{d}t\,(3x^2 - 1)\dot{x} = \int \mathrm{d}x\,(3x^2 - 1) = x^3 - x + C,$$

where C is the constant of integration. Thus, we have the relation:

$$\frac{1}{2}\dot{x}^2 = x^3 - x + C.$$

2. We plot the phase portrait diagram using Python. The result is given in Figure 1.

The equilibrium points occur where $\dot{x} = 0$ and $\ddot{x} = 0$. Setting $3x^2 - 1 = 0$ gives $x = \pm\frac{1}{\sqrt{3}}$. Thus, the coordinates of the two equilibrium points are:

$$\left(-\frac{1}{\sqrt{3}}, 0\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, 0\right).$$

This corresponds to the two extrema in the phase portrait diagram.

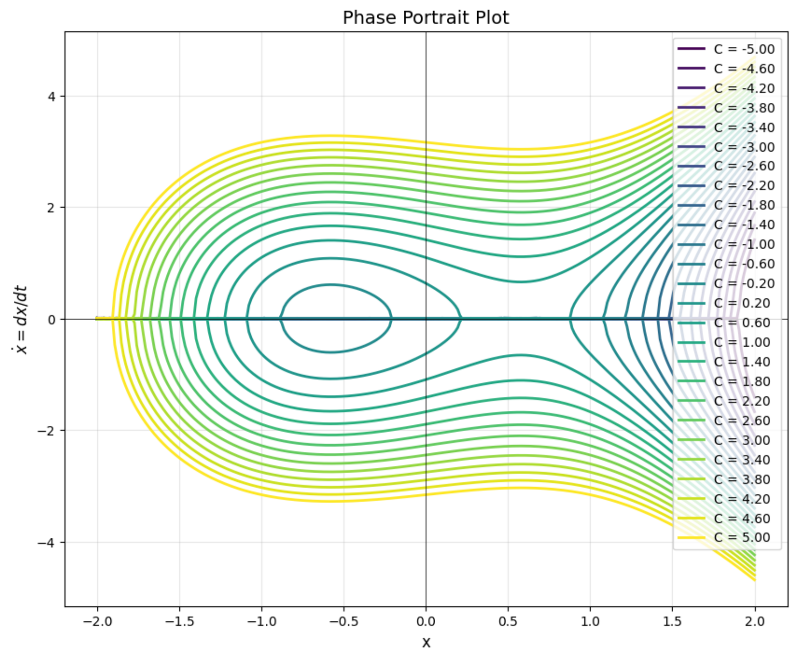

Figure 1: Phase portrait in the $(x, \dot{x})$-phase plane for various values of C.

Exercise 2. By sketching the curve $\dot{x} = X(x)$, determine the equilibrim points and corresponding domains of attraction when

1. $X(x) = x^2 - x - 2$
2. $X(x) = e^{-x} - 1$
3. $X(x) = \sin x$.

Now, check that this qualitative analysis is correct by actually solving each equation with initial conditions $x = x_0$ when $t = 0$. Which method do you think is easier to use, qualitative or quantitative?

Solution 2.

Steps:

1. Sketch the curves $\dot{x} = X(x)$ for each case using a plotting code.
2. Determine the equilibrim points and corresponding domains of attraction by looking at the slopes of the curves at the equilibrium points.
3. Solve each differential equation explicitly.

Method:

1. The phase portraits for each case are plotted using Python. The results are given in Figures 2, 3, and 4.
2. The equilibrim points are points where $\dot{x} = 0$. For 1., $x^2 - x - 2 = (x-2)(x+1)$, so the equilibrim points are $x = -1$ and $x = 2$. The domain of attraction is given by points where $X(x) < 0$ to the right of the equilibrim point and $X(x) > 0$ to the left. Thus, the domain of attraction is $(-\infty, 2)$. For 2., $e^{-x} - 1 = 0$ gives $x = 0$ as the only solution. The domain of attraction for $x = 0$ is $(-\infty, \infty)$. For 3., $\sin x = 0$ gives $x = n\pi$ for $n \in \mathbb{Z}$. The domain of attraction is $((2n)\pi, (2n+2)\pi)$ for $n \in \mathbb{Z}$.
3. We solves each differential equation explicitly below:

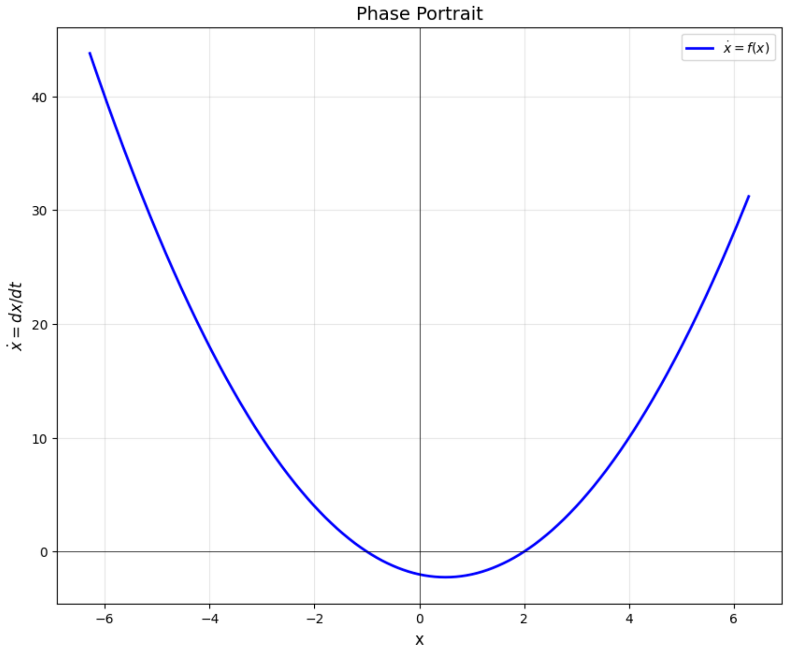

Figure 2: Phase portrait in the $(x, \dot{x})$-phase plane for $\dot{x} = x^2 - x - 2$.

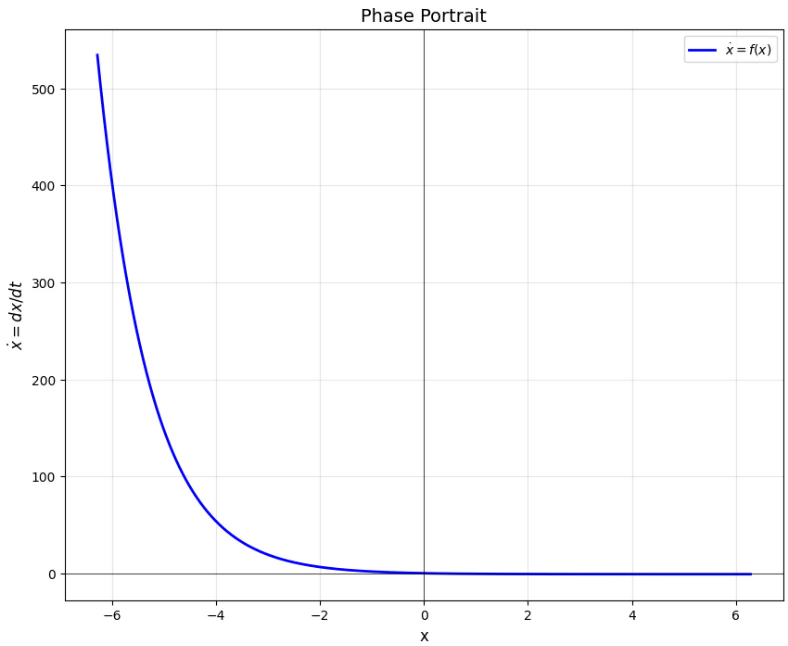

Figure 3: Phase portrait in the $(x, \dot{x})$-phase plane for $\dot{x} = e^{-x} - 1$.

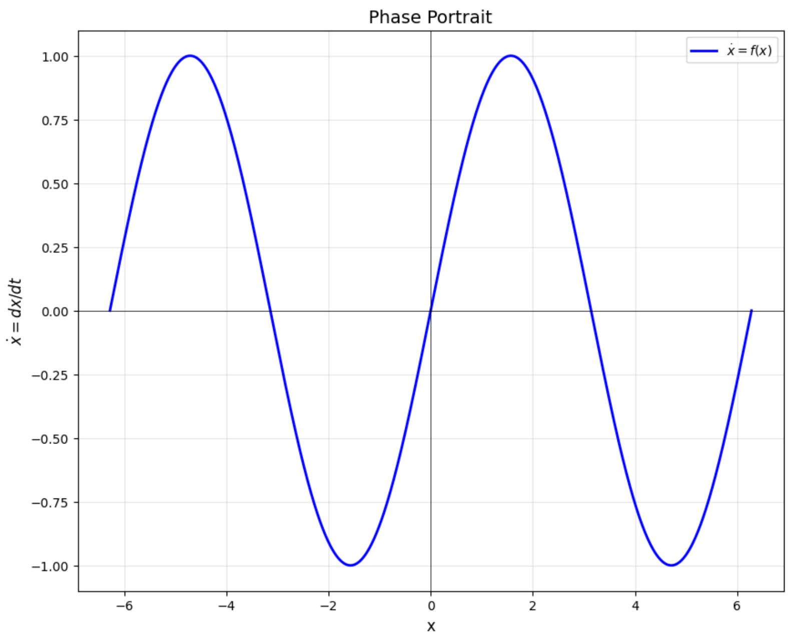

Figure 4: Phase portrait in the $(x, \dot{x})$-phase plane for $\dot{x} = \sin x$.

(1) For $X(x) = x^2 - x - 2$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) = x^2 - x - 2.$$

Separation of variables and partial fraction decomposition give

$$\int \frac{\mathrm{d}x}{x^2 - x - 2} = \int \mathrm{d}x \, \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right) = \int \mathrm{d}t$$

Then,

$$\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| = t + C_1$$

The initial condition $x(0) = x_0$ gives

$$C = \ln\left|\frac{x_0 - 2}{x_0 + 1}\right| \implies x = \frac{2 + C^{3t}}{1 - Ce^{3t}}.$$

Differentiating explicitly, we get

$$\dot{x} = X(x) = x^2 - x - 2 \implies x_{\text{equil}} = -1, 2.$$

When $x_0 < 2$, we have $\lim_{t\to\infty} x(t) = -1$, and when $x_0 > 2$, we have $\lim_{t\to\infty} x(t) = \infty$ since $0 < C < 1$, and the denominator approaches 0 in finite time. Therefore, the domain of attraction is $(-\infty, 2)$.

(2) For $X(x) = e^{-x} - 1$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) = e^{-x} - 1 \implies \int \frac{\mathrm{d}x}{e^{-x} - 1} = \int \mathrm{d}t.$$

Then,

$$-\ln|1 - e^x| = t + C_3 \implies x(t) = \ln\left(1 - C_4 e^{-t}\right), \quad C_4 = e^{-C_3}.$$

The initial condition $x(0) = x_0$ gives

$$t = \ln\left|\frac{1 - e^x}{1 - e^{x_0}}\right| \implies x = \ln\left(1 - (1 - e^{x_0})e^{-t}\right).$$

Differentiating explicitly, we get

$$\dot{x} = X(x) = e^{-x} - 1 \implies x_{\text{equil}} = 0.$$

When $x_0 < 0$, $e^{-x_0} - 1 < 0$, so $\dot{x} < 0$ and x is decreasing and bounded below by 0. When $x_0 > 0$, $e^{-x_0} - 1 > 0$, so $\dot{x} > 0$ and x is increasing and bounded above by 0. Solving $e^{-L} - 1 = 0$ gives $L = 0$, so the domain of attraction is $(-\infty, \infty)$.

(3) For $X(x) = \sin x$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) = \sin x \implies \int \frac{\mathrm{d}x}{\sin x} = \int \mathrm{d}t.$$

Then,

$$\ln\left|\tan\frac{x}{2}\right| = t + C_5 \implies x(t) = 2\arctan\left(C_6 e^t\right), \quad C_6 = e^{C_5}.$$

The initial condition $x(0) = x_0$ gives

$$x_0 = 2\arctan C_6 \implies C_6 = \tan\frac{x_0}{2},$$

and

$$x(t) = 2\arctan\left(\tan\frac{x_0}{2}e^t\right).$$

Differentiating explicitly, we get $\dot{x} = X(x) = \sin x$, so the equilibrim points are $x_{\text{equil}} = n\pi, n \in \mathbb{Z}$. When $(2n)\pi < x_0 < (2n+2)\pi$, $n\pi < \frac{x_0}{2} < (n+1)\pi$, and

$$0 < \tan\frac{x_0}{2} < \infty.$$

we have $\lim_{t\to\infty} x(t) = 2 \cdot \frac{\pi}{2} + 2n\pi = (2n+1)\pi$, where we are taking the principal value of the arctan function. Therefore, the domain of attraction is $((2n)\pi, (2n+2)\pi)$ for $n \in \mathbb{Z}$.

4. The qualitative method agrees with the quantitatvie method, and provides a quick overview of the system's behavior without explicit solutions, which is useful when the ODE does not admit simple closed-form solutions. Therefore, this method is easier to use in general.