2025 Fall Introduction to ODE

Homework 2 (Due Sep 15, 2025)

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Problem 1 (Legendre Polynomials). Let P_n be the Legendre polynomial of degree n. Prove that $|P_n'(x)| < n^2$ and $|P_n''(x)| < n^4$ for $-1 < x < 1$.

Solution 1. The Legendre polynomial $P_n(x)$ satisfies the Legendre differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{1}$$

We cite the paper *"On a question by D. I. Mendeleev". Zap. Imp. Akad. Nauk. St. Petersburg. 62: 1-24.* for the result referred to as **Markov brothers' inequality**: Let $P(x) \in \mathbb{R}[x]$ be a polynomial of degree n, then for each integer $k \geq 1$, we have

$$\max_{-1\leq x\leq 1}\left|P^{(k)}(x)\right| \leq \frac{n^2\left(n^2-1^2\right)\left(n^2-2^2\right)\cdots\left(n^2-(k-1)^2\right)}{1\cdot 3\cdot 5\cdots(2k-1)} \max_{-1\leq x\leq 1}|P(x)|\,. \tag{2}$$

Furthermore, we claim that for $x \in [-1,1]$,

$$|P(x)| \leq 1 \tag{3}$$

Proof. From *George Arfken et al. Mathematical Methods For Physicists, Second Edition, Academic Press (1970)* , we have the **Schläfli integral representation** of the Legendre polynomial:

$$P_n(z) = \frac{1}{2\pi i}\int_C \mathrm{d}\zeta\, \frac{(\zeta^2-1)^n}{2^n(\zeta-z)^{n+1}}\,, \tag{4}$$

where $z \in \mathbb{C}$ and C surrounds z. With the substitution $\zeta = z + \sqrt{z^2-1}e^{i\theta}$, we have

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\left(z+\sqrt{z^2-1}\,\cos\theta\right)^n\,. \tag{5}$$

This is the Laplace integral representation of the Legendre polynomial. For $x \in [-1,1]$, let $x = \cos\phi$, then

$$\begin{aligned} |P_n(\cos\phi)| &= \left|\frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\,(\cos\phi + i\sin\phi\cos\theta)^n\right| \\ &\leq \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\ |\cos\phi + i\sin\phi\cos\theta|^n \\ &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\ \left(\cos^2\phi + \sin^2\phi\cos^2\theta\right)^{\frac{n}{2}} \\ &\leq \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta = 1. \end{aligned} \tag{6}$$

□

By Markov brothers' inequality and equation (3), we have

$$\begin{aligned} P_n'(x) &\leq \max_{-1\leq x\leq 1}\left|P^{(k)}(x)\right| \leq n^2 \max_{-1\leq x\leq 1}|P(x)| = n^2, \\ P_n''(x) &\leq \max_{-1\leq x\leq 1}\left|P^{(k)}(x)\right| \leq \frac{n^2(n^2-1)}{3} \max_{-1\leq x\leq 1}|P(x)| = \frac{n^2(n^2-1)}{3} \leq n^4. \end{aligned} \tag{7}$$