2025 Fall Introduction to ODE

Homework 3 (Due Sep 22 12:00, 2025)

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Problem 1. Give the solutions, where possible in terms of the Bessel functions, of the differential equations

(a) $x\frac{d^2y}{dx^2} + (x+1)^2 y = 0$,

(b) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

Solution 1. Bessel's equation can be written in the form

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2-\nu^2)y = 0, \tag{1}$$

where ν is real and positive.

(a) We first solve the equation with the method of Frobenius. Multiply both sides by x to obtain

$$x^2\frac{d^2y}{dx^2} + x\,(x+1)^2\,y = 0.$$

Notice that $x(x+1)^2$ is analytic and $x=0$ is a singular point, so we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0.$$

Substituting into the differential equation, we find the equation

$$\begin{aligned} &r(r-1)a_0x^r + \left[r(r+1)a_1 + a_0\right]x^{r+1} + \left[(r+1)(r+2)a_2 + a_1 + 2a_0\right]x^{r+2} \\ &+ \sum_{n=3}^{\infty}\left[(n+r)(n+r-1)a_n + a_{n-1} + 2a_{n-2} + a_{n-3}\right]x^{n+r} = 0. \end{aligned} \tag{2}$$

The x^r terms gives $r=0$ or 1, but the x^{r+1} terms gives $a_1 = -\frac{a_0}{r(r+1)}$, which is undefined for $r=0$, so we must have $r=1$. Solving for the coefficients of a_0, a_1, and a_2, we find

$$a_1 = -\frac{1}{2}a_0, \quad a_2 = -\frac{1}{4}a_0. \tag{3}$$

Then from the recurrence relation

$$\sum_{n=3}^{\infty}\left[(n+r)(n+r-1)a_n + a_{n-1} + 2a_{n-2} + a_{n-3}\right]x^{n+r} = 0,$$

thus

$$a_n = -\frac{1}{n(n+1)}\left(a_{n-1} + 2a_{n-2} + a_{n-3}\right), \quad n \geq 3, \tag{4}$$

we can recursively solve for a_n in terms of a_0, giving the series solution

$$\begin{aligned} y(x) = a_0\Big[&x - \frac{1}{2}x^2 - \frac{1}{4}x^3 + \frac{1}{48}x^4 + \frac{47}{960}x^5 + \frac{17}{3200}x^6 + \frac{397}{134400}x^7 \\ &- \frac{2537}{2508800}x^8 + \frac{12091}{541900800}x^9 + \frac{2684597}{48771072000}x^{10} + \frac{44458303}{5364817920000}x^{11} + \cdots\Big]. \end{aligned} \tag{5}$$

The other linearly independent solution may be found using the reduction of order method. The above coefficients were verified with the following MATLAB code.

```
function a = gen_coeffs_rational(N)
    a=sym('a', [N+1, 1]);
    a(1:3)=[1;-1/2;-1/4]; % initial values
    for k=4:(N+1)
        n=sym(k-1);
        a(k)=simplify(-(a(k-1)+2*a(k-2)+a(k-3))/(n*(n+1)));
    end
end

N=10; a=gen_coeffs_rational(N);
fprintf('a_0 = %s\n', char(a(1)));
arrayfun(@(i) fprintf('a_%d = %s\n', i-1, char(a(i))), 2:N+1);
```

(b) This is Legendre's equation. The general solution is given by

$$y(x) = AP_n(x) + BQ_n(x),$$

where $P_n(x)$ and $Q_n(x)$ are the Legendre functions of the first and second kind, respectively, and A and B are constants. I will give a series expansion of $P_n(x)$ in terms of a Fourier-Bessel series on $[0, 1]$. First write

$$P_n(x) = \sum_{m=1}^{\infty} a_m J_l(\alpha_m x), \tag{6}$$

where $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ are roots of $J_l(x)$. Since $P_n(x)$ and $P_n'(x)$ are piecewise continuous on $[0, 1]$, the Fourier-Bessel series converges. Let $n, l \in \mathbb{Z}$, then the coefficients a_m are given by

$$\begin{aligned}
a_m &= \frac{1}{2\left[J_l'(\alpha_m)\right]^2} \int_{-1}^{1} \mathrm{d}x\, x P_n(x) J_l(\alpha_m x) \\
&= \frac{2^{n-1}}{\left[J_l'(\alpha_m)\right]^2} \sum_{k=1}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} \left[\int_0^1 \mathrm{d}x\, x^{k+1} J_l(\alpha_m x) + \int_{-1}^{0} \mathrm{d}x\, x^{k+1} J_l(\alpha_m x)\right] \\
&= \frac{2^{n-1}}{\left[J_l'(\alpha_m)\right]^2} \sum_{k=1}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} \left[\int_0^1 \mathrm{d}x\, x^{k+1} J_l(\alpha_m x) + (-1)^k \int_{1}^{0} \mathrm{d}x\, x^{k+1} J_l(-\alpha_m x)\right] \\
&= \frac{2^{n-1}}{\left[J_l'(\alpha_m)\right]^2} \sum_{k=1}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} \\
&\times \left(\left\{ \frac{2^{k+1}\Gamma\left(\frac{l+k+2}{2}\right)}{\alpha_m^{k+2}\Gamma\left(\frac{l-k}{2}\right)} + \alpha_m^{-(k+1)} \left[(k+l) J_l(\alpha_m) S_{k,l-1}(\alpha_m) - J_{l-1}(\alpha_m) S_{k,l-1}(\alpha_m)\right] \right\} \right. \qquad (7) \\
&+ \left. \left\{ \frac{2^{k+1}\Gamma\left(\frac{l+k+2}{2}\right)}{\alpha_m^{k+2}\Gamma\left(\frac{l-k}{2}\right)} + (-\alpha_m)^{-(k+1)} \left[(l+k) J_l(-\alpha_m) S_{k,l-1}(-\alpha_m) - J_{l-1}(-\alpha_m) S_{k+1,l}(-\alpha_m)\right] \right\} \right) \\
&= \frac{2^{n-1}}{\left[J_l'(\alpha_m)\right]^2} \sum_{k=1}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} \frac{2^{k+2}\Gamma\left(\frac{l+k+2}{2}\right)}{\alpha_m^{k+2}\Gamma\left(\frac{l-k}{2}\right)} \\
&+ \frac{1}{\alpha_m^{k+1}} \left\{ (k+l) \left[J_l(\alpha_m) S_{k,l-1}(\alpha_m) + (-1)^{l+k} J_{-l}(\alpha_m) S_{k,-l+1}(\alpha_m)\right] \right. \\
&- \left. \left[J_{l-1}(\alpha_m) S_{k+1,l}(\alpha_m) + (-1)^{l+k} J_{-l+1}(\alpha_m) S_{k+1,-l}(\alpha_m)\right] \right\},
\end{aligned}$$

where we have used the series expansion for $P_n(x)$

$$P_n(x) = 2^{n-1} \sum_{k=1}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} x^k, \tag{8}$$

the symmetry condition for $J_l(x)$

$$J_l(-x) = (-1)^l J_{-l}(x), \tag{9}$$

and the Bessel function integral identity from [Gradshteyn & Ryzhik] *Table of integrals, series, and products* 6.561-13.

$$\int_0^1 \mathrm{d}x\, x^\mu J_\nu(ax) = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+2}{2}\right)}{a^{\mu+1}\Gamma\left(\frac{\nu-\mu}{2}\right)} + \frac{1}{a^\mu}\left[(\mu+\nu)J_\nu(a)S_{\mu,\nu-1}(a) - J_{\nu-1}(a)S_{\mu+1,\nu}(a)\right]. \quad (10)$$

Here we have defined the Lommel function $S_{\mu,l}$ to be

$$\begin{aligned} S_{\mu,l}(x) &= \frac{\pi}{2}\left[Y_l(x)\int_0^x \mathrm{d}u\, u^\mu J_l(u) - J_l(x)\int_0^x \mathrm{d}u\, u^\mu Y_l(u)\right] \\ &+ 2^{\mu-1}\Gamma\left(\frac{\mu-l+1}{2}\right)\Gamma\left(\frac{\mu+l+1}{2}\right)\left[\sin\left(\frac{1}{2}(\mu-l)\pi\right)J_l(x) - \cos\left(\frac{1}{2}(\mu-l)\pi\right)Y_l(x)\right]. \end{aligned} \quad (11)$$

Note that this function is also known as $s^{(2)}_{\mu,l}$, while the first term inside square brackets is known as $s^{(1)}_{\mu,l}$.

For $l \in \mathbb{Z}$, the Lommel function $S_{\mu,l}(x)$ satisfies the symmetry identity

$$S_{\mu,l}(-x) = (-1)^{\mu+1}S_{\mu,-l}(x). \quad (12)$$

Finally, using the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}J_l(x) = \frac{1}{2}\left[J_{l-1}(x) - J_{l+1}(x)\right], \quad (13)$$

we have

$$\begin{aligned} a_m &= \frac{2^{n+1}}{J_{l-1}(2\alpha_m) - J_{l+1}(2\alpha_m)}\sum_{k=1}^{\infty}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}\frac{2^{k+2}\Gamma\left(\frac{l+k+2}{2}\right)}{\alpha_m^{k+2}\Gamma\left(\frac{l-k}{2}\right)} \\ &+ \frac{1}{\alpha_m^{k+1}}\left\{(k+l)\left[J_l(\alpha_m)S_{k,l-1}(\alpha_m) + (-1)^{l+k}J_{-l}(\alpha_m)S_{k,-l+1}(\alpha_m)\right]\right. \\ &- \left.\left[J_{l-1}(\alpha_m)S_{k+1,l}(\alpha_m) + (-1)^{l+k}J_{-l+1}(\alpha_m)S_{k+1,-l}(\alpha_m)\right]\right\}. \end{aligned} \quad (14)$$

Therefore, one solution is

$$\begin{aligned} P_n(x) &= 2^{n+1}\sum_{m=1}^{\infty}\frac{1}{J_{l-1}(\alpha_m) - J_{l+1}(\alpha_m)}\sum_{k=1}^{\infty}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}\frac{2^{k+2}\Gamma\left(\frac{l+k+2}{2}\right)}{\alpha_m^{k+2}\Gamma\left(\frac{l-k}{2}\right)} \\ &+ \frac{1}{\alpha_m^{k+1}}\left\{(k+l)\left[J_l(\alpha_m)S_{k,l-1}(\alpha_m) + (-1)^{l+k}J_{-l}(\alpha_m)S_{k,-l+1}(\alpha_m)\right]\right. \\ &- \left.\left[J_{l-1}(\alpha_m)S_{k+1,l}(\alpha_m) + (-1)^{l+k}J_{-l+1}(\alpha_m)S_{k+1,-l}(\alpha_m)\right]\right\}J_l(\alpha_m x). \end{aligned} \quad (15)$$

The other solution $Q_n(x)$ may be expressed in terms of $P_n(x)$, $\ln(x)$, and the standard recurrence relations for Legendre functions.

Problem 2. Determine the coefficients of the Fourier-Bessel series for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1, \\ -1 & \text{for } 1 \le x \le 2, \end{cases}$$

in terms of the Bessel function $J_0(x)$.

Solution 2. Suppose the Fourier-Bessel series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\alpha_n x), \quad (16)$$

where $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ are roots of $J_0(2x)$. Since $f(x)$ and $f'(x) = 0$ (except at $x = 0$) are piecewise continuous on $[0, 2]$, the Fourier-Bessel series converges. First we compute two integrals using the identity $J_0'(2\alpha_n) = \frac{1}{2} J_1(2\alpha_n)$ and $\frac{\mathrm{d}}{\mathrm{d}u}(uJ_1(u)) = uJ_0(u)$:

$$\begin{aligned}\int_0^1 \mathrm{d}x\, x J_0(\alpha_n x) &= \frac{1}{\alpha_n^2} \int_0^{\alpha_n} \mathrm{d}u\, u J_0(u) \\ &= \frac{1}{\alpha_n^2} \left[uJ_1(u)\right]_0^{\alpha_n} \\ &= \frac{1}{\alpha_n} J_1(\alpha_n),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\int_1^2 \mathrm{d}x\, x J_0(\alpha_n x) &= \frac{1}{\alpha_n^2} \int_{\alpha_n}^{2\alpha_n} \mathrm{d}u\, u J_0(u) \\ &= \frac{1}{\alpha_n^2} \left[uJ_1(u)\right]_{\alpha_n}^{2\alpha_n} \\ &= \frac{1}{\alpha_n^2} \left(2\alpha_n J_1(2\alpha_n) - \alpha_n J_1(\alpha_n)\right) \\ &= \frac{1}{\alpha_n} \left(2J_1(2\alpha_n) - J_1(\alpha_n)\right),\end{aligned} \tag{18}$$

where we have use a substitution $u = \alpha_n x$. Then, the coefficients a_n are given by

$$\begin{aligned}a_n &= \frac{1}{\left[2J_1(2\alpha_n)\right]^2} \int_0^2 \mathrm{d}x\, x f(x) J_0(\alpha_n x) \\ &= \frac{1}{\left[2J_1(2\alpha_n)\right]^2} \left(-\int_1^2 \mathrm{d}x\, x J_0(\alpha_n x) + \int_0^1 \mathrm{d}x\, x J_0(\alpha_n x)\right) \\ &= \frac{1}{2\alpha_n \left[J_1(2\alpha_n)\right]^2} \left(-2J_1(2\alpha_n) + J_1(\alpha_n) + J_1(\alpha_n)\right), \\ &= \frac{1}{\alpha_n \left[J_1(2\alpha_n)\right]^2} \left(J_1(\alpha_n) - J_1(2\alpha_n)\right).\end{aligned} \tag{19}$$

Therefore,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n \left[J_1(\alpha_n)\right]^2} \left(J_1(\alpha_n) - J_1(2\alpha_n)\right) J_0(\alpha_n x). \tag{20}$$