2025 Fall Introduction to ODE

Homework 4 (Due Sep 29 12:00, 2025)

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Problem 1. Comment on the difficulties that you face when trying to construct the Green's function for the boundary value problem

$$y''(x) + y(x) = f(x) \quad \text{subject to} \quad y(a) = y'(b) = 0. \tag{1}$$

Solution 1.

Steps:

1. Construct the general solution to the homogeneous equation $y'' + y = 0$.
2. Solve for the Green's function $G(x, \xi)$ using the boundary conditions and the jump condition at $x = \xi$.
3. Write the solution to the inhomogeneous equation as

$$y(x) = \int_a^b G(x, \xi) f(\xi) \, \mathrm{d}\xi, \tag{2}$$

and discuss the problem found.

Method: First let's pick the standard fundamental solutions to the homogeneous case $y''+y = 0$ such that they satisfy the boundary conditions. We have:

$$\begin{aligned} y_1(x) &= \sin(x - a), \quad y_1(a) = 0, \\ y_2(x) &= \cos(x - b), \quad y_2'(b) = 0. \end{aligned} \tag{3}$$

The Wronskian is a constant given by

$$W = y_1 y_2' - y_2 y_1' = -\cos(a - b), \tag{4}$$

and the Green's function is given by

$$G(x, t) = \begin{cases} [y_1(x) y_2(\xi)] / W, & a \le x < \xi \le b, \\ [y_1(\xi) y_2(x)] / W, & a \le \xi < x \le b, \end{cases} \tag{5}$$

whenever $W(x)$ is nonzero. However, if $a - b = n\pi/2$, $n \in \mathbb{Z}$, then $W = 0$ and the Green's function cannot be constructed. We can interpret this result by notiving that the boundary conditions are not independent when $a - b = n\pi/2$, since $W = 0$ at these points. In this case, the boundary value problem may not have a solution for arbitrary $f(x)$.

Problem 2. Write the generalized Legendre equation,

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left\{ n(n+1) - \frac{m^2}{1 - x^2} \right\} y = 0, \tag{6}$$

as a Sturm-Liouville equation.

Solution 2. Steps:

1. Rewrite the equation in the standard form of a Sturm-Liouville problem.
2. Identify the functions $p(x), q(x), r(x)$ and the eigenvalue λ.
3. Discuss the boundary conditions at the endpoints $x = \pm 1$.

Method: Sturm-Liouville equations are of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) = -\lambda r(x)y(x) = 0, \tag{7}$$

where λ is the eigenvalue which depends on the boundary conditions. Furthermore, the functions $p(x), q(x), r(x)$ are real-valued and continuous on the closed interval $[a, b]$, $p(x)$ is differentiable, and $p(x) > 0, r(x) > 0$ on the open interval (a, b).

From the textbook, the endpoint $x = a$ is a **singular endpoint** if $a = -\infty$ or if $a < \infty$ but the above conditions do not hold on the closed interval $[a, c]$ for some $c \in (a, b)$. Similar definitions hold for the other endpoint, $x = b$. Hence ± 1 are singular endpoints of the generalized Legendre equation. Therefore, we apply Friedrichs boundary conditions.

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right] - \left(\frac{m^2}{1-x^2}\right)y(x) = -n(n+1)y(x), \tag{8}$$

which is a Sturm-Liouville equation with eigenvalue $\lambda = n(n+1)$ and functions

$$p(x) = 1 - x^2, \quad q(x) = -\frac{m^2}{1-x^2}, \quad r(x) = 1 \tag{9}$$

satisfying the conditions above on the interval $(-1, 1)$. Since the Sturm-Liouville operator is singular at ± 1, we assume Friedrichs boundary conditions following the description in [King & Billingham & Otto]. That is, we require that the solution $y(x, \lambda)$ satisfies

$$|y(x,\lambda)| \leq A \text{ for } x \in (-1, 0] \text{ and } |y(x,\lambda)| \leq B \text{ for } x \in [0, 1), \tag{10}$$

for some $A, B \in \mathbb{R}_{\geq 0}$.

Problem 3. Show that

$$-(xy'(x))' = \lambda x y(x), \tag{11}$$

is self-adjoint on the interval $(0, 1)$, with $x = 0$ a singular endpoint and $x = 1$ a regular endpoint with the condition $y(1) = 0$.

Solution 3.

Steps:

1. Rewrite the equation in the standard form of a Sturm-Liouville problem.
2. Use Lagrange's identity to show that L is self-adjoint

Method: A linear operator is said to be self-adjoint if it satisfies

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \tag{12}$$

where the inner product is defined as

$$\langle u, v\rangle = \int_0^1 \mathrm{d}x\, u(x)v(x). \tag{13}$$

Expanding the left-hand side, we can write the differential equation in terms of a linear operator L:

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \lambda x y(x) \equiv Ly = 0. \tag{14}$$

Let u, v be two functions satisfying the boundary conditions. Then, by Lemma 4.1 (Lagrange's identity) in [King & Billingham & Otto], let

$$L = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x} + \lambda x \tag{15}$$

be a linear differential operator on (0,1), and $u, v \in C^2(0,1)$, then

$$u(Lv) - v(Lu) = \left[p\left(uv' - u'v\right)\right], \tag{16}$$

thus

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \left[x\left(u(x)v'(x) - u'(x)v(x)\right)\right]_0^1 = 0. \tag{17}$$

where the terms at $x = 1$ vanish due to the boundary condition $y(1) = 0$, and the terms at $x = 0$ vanish by some additional regularity condition on $y(x)$ at the singular endpoint. One sufficient and natural choice would be

$$\lim_{x\to 0^+} xy'(x) = 0. \tag{18}$$

Therefore, we have shown that

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \tag{19}$$

and L is self-adjoint.