2025 Fall Introduction to ODE

Homework 5 (Due Oct 6 12:00, 2025)

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Problem 1. Using Laplace transforms, solve the initial value problem

$$\frac{dx}{dt} + x + \frac{dy}{dt} = 0, \quad \frac{dx}{dt} - x + 2\frac{dy}{dt} = e^{-t}, \quad \text{subject to } x(0) = y(0) = 1.$$

Solution 1.

Steps:

1. Take the Laplace transform of both equations.
2. Solve the resulting algebraic equations for $X(s)$ and $Y(s)$.
3. Find the inverse Laplace transforms to get $x(t)$ and $y(t)$.

Method: Recall that the Laplace transform of derivatives is given by

$$\mathcal{L}[f'] = sF(s) - f(0), \quad \mathcal{L}[f''] = s^2F(s) - sf(0) - f'(0). \tag{1}$$

Write the Laplace transform of $x(t)$ and $y(t)$ as $X(s)$, $Y(s)$, respectively. Then we have

$$\begin{aligned} sX(s) - x(0) + X(s) + sY(s) - y(0) &= 0, \\ sX(s) - x(0) - X(s) + 2sY(s) - 2y(0) &= \frac{1}{s+1}. \end{aligned}$$

Substitute the initial conditions $x(0) = y(0) = 1$ into the equations to get

$$\begin{aligned} (s+1)X(s) + sY(s) &= 2, \\ (s-1)X(s) + 2sY(s) &= \frac{1}{s+1} + 3, \end{aligned}$$

so

$$\begin{aligned} X(s) &= \frac{s}{(s+1)(s+3)} = \frac{3}{2(s+3)} - \frac{1}{2(s+1)}, \\ Y(s) &= \frac{s+6}{s(s+3)} = -\frac{1}{s+3} + \frac{2}{s}. \end{aligned}$$

Therefore, the inverse Laplace transforms are given by

$$\begin{aligned} x(t) &= \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t}, \\ y(t) &= -e^{-3t} + 2. \end{aligned}$$

Problem 2. Show that $\mathcal{L}[tf(t)] = -\frac{dF}{ds}$, where $F(s) = \mathcal{L}[f(t)]$. Hence solve the initial value problem

$$\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - 4x = 1, \quad \text{subject to } x(0) = x'(0) = 0.$$

Solution 2.

Steps:

1. We prove the desired property with integration by parts.
2. Transform the differential equation using the Laplace transform into a first-order ODE.
3. Solve the first-order ODE for $X(s)$.
4. Find the inverse Laplace transform to get $x(t)$.

Method: By definition of the Laplace transform and Leibniz' rule of differentiating under the integral sign, we have

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{d}t\, e^{-st}f(t) \\ &= \int_0^\infty \mathrm{d}t\, \frac{\partial}{\partial s}\left(e^{-st}f(t)\right) \\ &= -\int_0^\infty \mathrm{d}t\, e^{-st}tf(t) \\ &= \mathcal{L}[tf(t)]. \end{aligned} \tag{2}$$

Taking the Laplace transform of both sides of the differential equation using 1 and 2, we have

$$s^2X(s) - sx(0) - x'(0) - 2\frac{\mathrm{d}}{\mathrm{d}s}\left(sX(s) - x(0)\right) - 4X(s) = \frac{1}{s},$$

$$\implies s^2X(s) - 2sX'(s) - 6X(s) = \frac{1}{s}.$$

Rearranging gives

$$X'(s) + \frac{6-s^2}{2s}X(s) = -\frac{1}{2s^2}.$$

This is a first-order linear ODE. The integrating factor is given by

$$\mu(s) = e^{\int \frac{6-s^2}{2s}\mathrm{d}s} = s^3e^{-\sqrt{s/4}}.$$

Therefore, we have

$$X(s)\mu(s) = -\frac{1}{2}\int_0^s \mathrm{d}s'\, \frac{1}{(s')^2}(s')^3e^{-(s')^2/4} = e^{-(s')^2/4} - 1,$$

where we used a substitution $u = (s')^2/4$ to evaluate the integral. Thus, we have

$$X(s) = \frac{e^{-s^2/4} - 1}{s^3e^{-s^2/4}} = \frac{1 - e^{s^2/4}}{s^3}.$$

The inverse Laplace transform can be found by noting that

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{e^{s^2/4}}{s^3}\right].$$

The first term is given by

$$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2},$$

while the second term tends to infinity as $s \to \infty$ and thus does not have a well-defined inverse Laplace transform. Therefore, the solution to the initial value problem is simply

$$x(t) = \frac{t^2}{2}.$$

Problem 3. Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s-1)(s^2+1)},$$

by (a) expressing $F(s)$ as partial fractions and inverting the constituent parts, and (b) using the convolution theorem.

Solution 3.

Steps:

1. Express $F(s)$ as partial fractions:
2. Invert the constituent parts using known Laplace transforms.
3. Use the convolution theorem to directly find the inverse Laplace transform.

Method:

(a) We can write

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{(A+B)s^2 + (C-B)s + (A-C)}{(s-1)(s^2+1)}.$$

and solve for A, B, C. This gives

$$A = \frac{1}{2}, \quad B = C = -\frac{1}{2}.$$

Therefore, we have

$$F(s) = \frac{1/2}{s-1} - \frac{1/2s}{s^2+1} - \frac{1/2}{s^2+1}.$$

Apply the inverse Laplace transform to each term to get

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2}e^t - \frac{1}{2}\sin t - \frac{1}{2}\cos t = \frac{1}{2}\left(e^t - \sin t - \cos t\right).$$

(b) Next, we use the convolution theorem to directly find the inverse Laplace transform. The convolution theorem states that given three functions $f(t), g(t), h(t)$ and their respective Laplace transforms $F(s), G(s), H(s)$ satisfying $F(s) = G(s)H(s)$, then

$$\mathcal{L}^{-1}[F(s)] = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Equivalently, we can write it with the convolution operation:

$$f(t) = (g * h)(t).$$

Let $G(s) = (s-1)^{-1}$ and $H(s) = (s^2+1)^{-1}$. Then we have $g(t) = e^t$ and $h(t) = \sin(t)$. Therefore, we can write via the convolution theorem that

$$f(t) = \int_0^t g(t-\tau)h(\tau)d\tau = \int_0^t \mathrm{d}\tau\, e^{t-\tau}\sin(\tau).$$

The integral can be evaluated using integration by parts, yielding

$$f(t) = \int_0^t e^{t-\tau}\sin(\tau)d\tau = -e^{-\tau}\left(\sin\tau + \cos\tau\right)\Big|_0^t - f(t),$$

so

$$f(t) = \frac{1}{2}\left(e^t - \sin t - \cos t\right).$$