2025 Fall Introduction to ODE

Homework 6 (Due Oct 27 12:00, 2025)

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Problem 1. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ satisfy $a_{ij} = \frac{i}{j}$ for $i, j = 1, \cdots, n$. Calculate e^{At} for $t > 0$.

Solution 1.

Steps:

1. Find formula for A^n using induction.
2. Use the Taylor series to find e^{At}.

Method:

Notice that

$$\begin{aligned} A^2 &= \left(a_{ij}^2\right) = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} \frac{i}{k} \cdot \frac{k}{j} = \sum_{k=1}^{n} \frac{i}{j} = n \cdot \frac{i}{j} = nA, \\ A^3 &= A^2 \cdot A = nA \cdot A = n^2 A, \\ &\vdots \\ A^k &= n^{k-1}A. \end{aligned}$$

By induction, suppose $A^k = n^{k-1}A$ holds for some $k \geq 1$. Then

$$A^{k+1} = A^k \cdot A = n^{k-1}A \cdot A = n^{k-1} \cdot nA = n^k A.$$

Thus, by induction, we have $A^k = n^{k-1}A$ for all $k \geq 1$.

Let's compute the matrix exponential e^{At} using its Taylor series expansion:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots = I + \sum_{k=1}^{\infty} \frac{(At)^k}{k!}.$$

Substituting $A^k = n^{k-1}A$ into the equation, we have

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{(n^{k-1}At^k)}{k!} = I + \frac{A}{n} \sum_{k=1}^{\infty} \frac{(nt)^k}{k!} = I + \frac{A}{n}\left(e^{nt} - 1\right).$$

Problem 2. Suppose there is a constant K such that a fundamental matrix solution X of the real system $\dot{x} = A(t)x$ satisfies $|X(t)| \leq K$, $t \geq \beta$ and

$$\liminf_{t \to \infty} \int_{\beta}^{t} \operatorname{tr} A(s)ds > -\infty.$$

Prove that X^{-1} is bounded on $[\beta, \infty)$ and no nontrivial solution of $\dot{x} = A(t)x$ approaches zero as $t \to \ infty$.

Solution 2.

Steps:

1. Show that $\det X$ is bounded from below.
2. Show that X^{-1} is bounded on $[\beta, \infty)$.
3. Show that no nontrivial solution approaches zero as $t \to \infty$.

   Method:

   By lemma 1.5 (Liouville's formula) of Hale's Ordinary Differential Equations, we have

$$\det X(t) = \det X(\beta) \exp\left(\int_\beta^t \operatorname{tr}(A(s))ds\right), \quad t \geq \beta.$$

Let $I(t) \equiv \liminf_{t\to\infty} \int_\beta^t \operatorname{tr}(A(s))ds$. Since $I(t) > -\infty$, there exists some constant $m > 0$ and $T \geq t$ such that

$$I(t) \geq -m, \quad t \geq T.$$

Let $m_0 = \min\{\inf_{\beta\leq t\leq T} I(t), m, \}$, then Liouville's formula gives

$$|X(t)| = |X(\beta)|e^{I(t)} \geq |X(\beta)|e^{-m_0} \equiv A, \quad t \geq \beta.$$

In finite dimensional vector spaces, all vector norms are equivalent. Treat $X(t)$ as an element of the vector space $M_n(\mathbb{R})$, the boundedness property $|X(t)| \leq K$ for $t \geq \beta$ implies that there exists some constant $B_K > 0$ such that

$$\max |x_{ij}(t)| \leq B_K, \quad t \geq \beta,$$

where the left hand side is the max norm. Hence $|x_{ij}(t)|$ is bounded for all $i, j = 1, \cdots, n$ and $t \geq \beta$, and every minor of $X(t)$ is bounded uniformly for $t \geq \beta$. We have $|\operatorname{adj} X(t)| \leq C$ for $t \geq \beta$, and

$$|X^{-1}(t)| = \frac{|\operatorname{adj} X(t)|}{|X(t)|} \leq \frac{C}{A}, \quad t \geq \beta,$$

so X^{-1} is bounded on $[\beta, \infty)$. Since every nontrivial solution of $\dot{x} = A(t)x$ can be expressed as $x(t) = X(t)c$ for some nonzero constant vector c, suppose there exists a nontrivial solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$. Then

$$|c| = |X^{-1}(t)x(t)| \leq |X^{-1}(t)| \cdot |x(t)| \to 0, \quad t \to \infty,$$

which is a contradiction. Therefore, no nontrivial solution of $\dot{x} = A(t)x$ approaches zero as $t \to \infty$.