2025 Fall Introduction to ODE

Homework 7 (Due Nov 3 12:00, 2025)

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Exercise 1. Let $A(t)$ and $B(t)$ be defined as

$$A(t) = \begin{pmatrix} -a & 0 \\ 0 & \sin\log t + \cos\log t - 2a \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ e^{-at} & 0 \end{pmatrix}, \quad t \geq 0,$$

where $1 < 2a < 1 + e^{-\pi}$. Is the system $\dot{x} = [A(t) + B(t)]x$ unstable? Prove or disprove your answer.

Solution 1.

Steps:

1. State the definition for a solution to be (Lyapunov) unstable.
2. Analyze the system $\dot{x} = [A(t) + B(t)]x$ and find its solution.
3. Find a lower bound for the solution for some specific initial condition and time sequence.
4. Show the sequence grows without bound, and conclude the zero solution is not Lyapunov stable.

Method:

1. A system is said to be unstable if there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there exists an initial condition $x(t_0)$ with $|x(t_0)| < \delta$ and a time $t > t_0$ such that $|x(t)| > \varepsilon$.
2. The matrix $A(t) + B(t)$ is lower-triangular, so we may directly solve for $x_1(t)$:

$$\dot{x} = [A(t) + B(t)]\, x = \begin{pmatrix} -a & 0 \\ e^{-at} & \sin(\log t) + \cos(\log t) - 2a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

First, we have $\dot{x}_1 = -a x_1$, which gives the solution $x_1(t) = x_1(0) e^{-at}$. Then substitute into the second equation:

$$\dot{x}_2 = x_1(0) e^{-at} + [\sin(\log t) + \cos(\log t) - 2a]\, x_2.$$

$$\implies \dot{x}_2 - (\sin\log t + \cos\log t - 2a)\, x_2 = x_1(0) e^{-at}.$$

This is a first-order linear ODE, so we can use the integrating factor method. The integrating factor is given by

$$\mu(t) = \exp\left(- \int [\sin(\log t) + \cos(\log t) - 2a]\, \mathrm{d}t \right) = e^{2at} e^{\sin(\log t)},$$

where we used $\frac{\mathrm{d}}{\mathrm{d}t}(t \sin\log t) = \sin\log t + \cos\log t$. Then, we have

$$x_2(t) = \frac{1}{\mu(t)} \int_0^t \mathrm{d}s \, \left(x_1(0) e^{-s \sin\log s} \right) = x_1(0) e^{t(\sin\log t - 2a)} \int_0^t \mathrm{d}s \, e^{-s \sin\log s}.$$

Since we only have to find one solution, we set $x_2(0) = 0$ for our following discussion.

Let $t_n = e^{2\pi n + \frac{\pi}{2}}$, then $\sin\log t_n = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$. Similarly, let $\tilde{t}_n = t_n e^{-\pi} = e^{2\pi n - \frac{\pi}{2}}$, so that $\sin\log \tilde{t}_n = \sin\left(2\pi n - \frac{\pi}{2}\right) = -1$. For each $n \in \mathbb{N}$ and $\xi \in (0,1)$, by continuity of snie function, there exists $\delta > 0$ such that $\sin x \leq -1 + \xi$ whenever $\xi \in \left[2\pi n - \frac{\pi}{2} - \delta, 2\pi n - \frac{\pi}{2} + \delta\right]$. Therefore, $\sin\log s \leq -1 + \xi$ whenever $\tilde{t}_n e^{-\delta} \leq s \leq \tilde{t}_n e^{\delta}$.

3. Let $S_n = \left[e^{\tilde{t}_n+\delta}, e^{\tilde{t}_n-\delta}\right]$, then we have $\sin\log s \le -1+\xi$ for all $s \in S_n$. Thus,

$$e^{-s\sin\log s} \ge e^{s(1-\xi)}, \quad s \in S_n.$$

$$\implies \int_0^t \mathrm{d}s\, e^{-s\sin\log s} \ge \int_{S_n} \mathrm{d}s\, e^{s(1-\xi)} \ge \tilde{t}_n \left(e^{\delta} - e^{-\delta}\right) e^{(1-\xi)\tilde{t}_n e^{-\delta}} \ge 0, \quad t \ge e^{\tilde{t}_n+\delta}.$$

Evaluate $x_2(t)$ at t_n gives

$$\begin{aligned} x_2(t_n) &\ge x_1(0) e^{t_n(1-2a)} \tilde{t}_n \left(e^{\delta} - e^{-\delta}\right) e^{(1-\xi)\tilde{t}_n e^{-\delta}} \\ &= x_1(0) \left(e^{\delta} - e^{-\delta}\right) t_n e^{-\pi} e^{t_n\left[(1-2a)+(1-\xi)e^{-\pi}e^{-\delta}\right]}. \end{aligned}$$

4. We have $1 < 2a < 1 + e^{-\pi}$, so $1 - 2a + e^{-\pi} > 0$. Consider the function $f(\xi,\delta) = (1-2a) + (1-\xi)e^{-\pi}e^{-\delta}$, by assumption $f(0,0) > 0$. Since f is continuous, there exists a disk of radius C about $(0,0)$ such that $f\left(\overline{\xi},\overline{\delta}\right) > 0$ for all $\overline{\xi},\overline{\delta}$ in the disk. Thus, we can choose $\xi \in (0,\overline{\xi})$ and $\delta = \min\{\delta', \overline{\delta}\}$, where δ' is the bound given earlier by the continuity of sine. Then $t_n\left[(1-2a)+(1-\xi)e^{-\pi}e^{-\delta}\right] > 0$, and $x_2(t) \to \infty$ as $n \to \infty$. Moreover, since $x_1(0)$ is bounded, the norm $\|x(t)\| \to \infty$ as $t \to \infty$.

Exercise 2. Consider the ODE system

$$\dot{x} = A(t)x + f(t,x), \tag{1}$$

where $A(t) \in \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continous and satisfies $|f(t,x)| \le C(t)|x|$, for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Here, $C(t)$ is a continous function satisfying

$$\int_t^{t+1} C(s)\,ds \le \gamma, \quad t \ge \beta,$$

for some constant $\gamma = \gamma(\beta) > 0$. Suppose the ODE system $\dot{x} = A(t)x$ is uniformly asymptotically stable with respect to the zero solution. Prove that there is a constant $r > 0$ such that the zero solution of 1 is uniformly asymptotically stable if $r > \gamma$.

Solution 2.

Steps:

1. State the definition of being uniformly asymptotically stable with respect to the zero solution.
2. Show the Duhamel property.
3. Bound the solution $x(t)$ using Gronwall's inequality to prove the claim.

Method:

1. A system is said to be uniformly asymptotically stable with respect to the zero solution if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any initial condition $|x(t_0)| < \delta$, the solution $x(t)$ satisfies $|x(t)| < \varepsilon$ for all $t \ge t_0$.
2. We first prove a proposition.

Proposition 1 (Duhamel's property)**.** Let $x(t)$ be the solution to the non-homogeneous system $\dot{x} = A(t)x + f(t,x)$, $t \ge t_0$. Then, the solution can be expressed as

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^t \mathrm{d}s\, \Phi(t,s) f(s,x(s)),$$

where $\Phi(t,t_0) = X(t)X(t_0)^{-1}$ is the state transition matrix of $\dot{x} = A(t)x$.

Proof. Let $y(t) = \Phi(t_0, t)x(t)$. Then we have $\Phi(t_0, t)X(t) = X(t_0)$, so

$$(\partial_t \Phi(t_0, t))\, X(t) + \Phi(t_0, t)\, (\partial_t X(t)) = 0.$$

$$\implies \partial_t \Phi(t_0, t) = -\Phi(t_0, t)\, (\partial_t X(t))\, X(t)^{-1} = -\Phi(t_0, t)A(t).$$

Thus, we have

$$\dot{y}(t) = \Phi(t_0, t)\, (\dot{x} - A(t)x) = \Phi(t_0, t) f(t, x(t)).$$

Integrating from t_0 to t, we get

$$y(t) = y(t_0) + \int_{t_0}^{t} \mathrm{d}s\, \Phi(t, s) f(s, x(s)) = \Phi(t_0, t)x(t_0) - x(t_0) = \int_{t_0}^{t} \mathrm{d}s\, \Phi(t_0, s) f(s, x(s)).$$

$$\implies x(t) = \Phi(t, t_0) \left[x(t_0) + \int_{t_0}^{t} \mathrm{d}s\, \Phi(t_0, s) f(s, x(s)) \right] = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \mathrm{d}s\, \Phi(t, s) f(s, x(s)),$$

since $\Phi(t, t_0)\Phi(t_0, s) = \Phi(t, s)$ by the semigroup property. □

3. For a linear time0varying system, uniform asymptotic stability of the zero solution is equivalent to uniform exponential stability. Thus, there exist positive constants K and r such that the state transition matrix $\Phi(t, t_0)$ satisfies

$$\|\Phi(t, t_0)\| \leq K e^{-\alpha(t - t_0)}, \quad t \geq t_0.$$

Using the Duhamel property, we have

$$\begin{aligned} \|x(t)\| &\leq \|\Phi(t, t_0)x(t_0)\| + \left\| \int_{t_0}^{t} \mathrm{d}s\, \Phi(t, s) f(s, x(s)) \right\| \\ &\leq K e^{-r(t - t_0)} \|x(t_0)\| + \int_{t_0}^{t} \mathrm{d}s\, K e^{-r(t - s)} C(s) \|x(s)\|. \end{aligned}$$

Let $u(t) = e^{rt}\|x(t)\|$, then

$$\|u(t)\| \leq K \left[e^{rt_0} \|x(t_0)\| + \int_{t_0}^{t} \mathrm{d}s\, C(s) e^{rs} \|x(s)\| \right] = K \left[u(t_0) + \int_{t_0}^{t} \mathrm{d}s\, C(s) u(s) \right].$$

Then

$$u(t) \leq K u(t_0) + K \int_{t_0}^{t} \mathrm{d}s\, C(s) u(s) \leq K u(t_0) \exp\left(\int_{t_0}^{t} \mathrm{d}s\, C(s) \right).$$

We can bound the term in the exponential using the assumption on $C(t)$:

$$\begin{aligned} \|x(t)\| &\leq K e^{-r(t - t_0)} \|x(t_0)\| \exp\left(\int_{t_0}^{t} \mathrm{d}s\, C(s) \right) \leq K e^{-r(t - t_0)} e^{\gamma(t - t_0 + 1)} \|x(t_0)\| \\ &= K e^{\gamma} e^{-(r - \gamma)(t - t_0)} \|x(t_0)\|. \end{aligned}$$

Since $r > \gamma$, we have $\|x(t)\| \to 0$ as $t \to \infty$. More precisely, for any $\varepsilon > 0$, let $\delta = \frac{1}{K} e^{-\gamma} \varepsilon$, then

$$\|x(t)\| = K e^{\gamma} e^{-(r - \gamma)(t - t_0)} \|x(t_0)\| < K e^{\gamma} \delta = \varepsilon$$

whenever $\|x(t_0)\| < \delta$. Thus, the zero solution of the system is uniformly asymptotically stable.