Application of Field Theory to Percolation Processes

Based on Hans-Karl Janssen, Uwe C. Tauber (2018). *The field theory approach to percolation. For the Quantum Field Theory II Course*

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What is Percolation and Why?

- Introduced by Broadbent & Hammersley (1957) to model fluid flow through porous media
- One of the simplest systems exhibiting continuous phase transition
- Percolating systems constitute universality classes, which can be studied with RG

Rmk. Percolation theory = 滲流理論

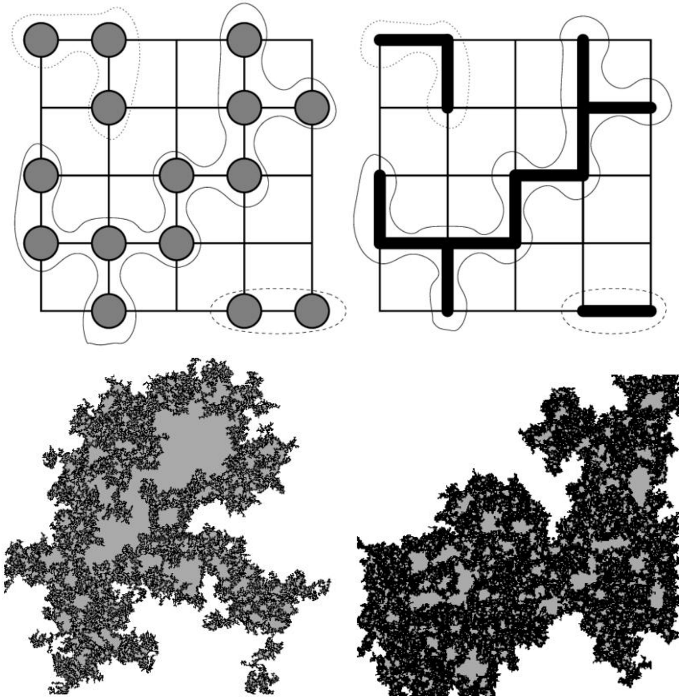


Fig. Site vs. bond percolation. Percolation is the emergence of an infinite connected component (a cluster) containing 0

Romer, Rudolf A. (2001). *Percolation, Renormalization and the Quantum-Hall Transition*

A Simple Example

- Infinitely many Bernoulli iid's with probability p on a 2d square lattice grid $\mathbb{Z}^2$
- When does an infinite cluster appear*?

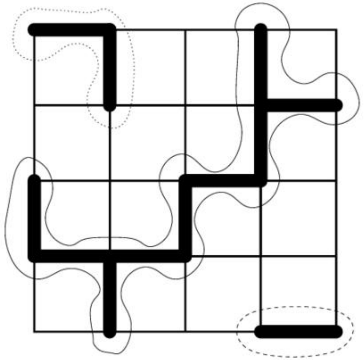
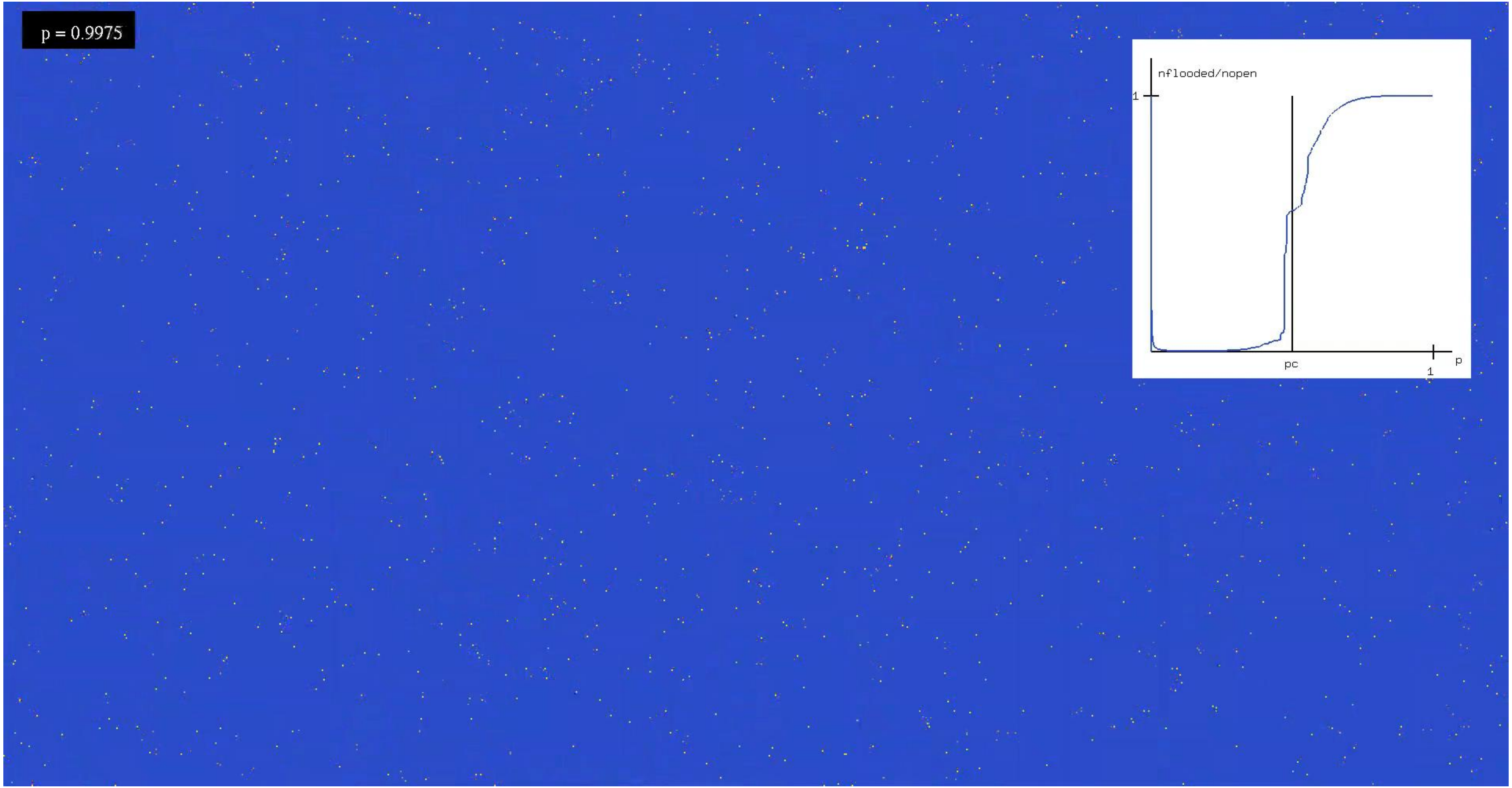


Fig. Bond percolation on square grids: probability p of being connected and $1-p$ of being not connected

iid: **i**ndependent and **i**dentically **d**istributed random variables

*: The event occurs with positive probability



Nils Berglund @YouTube

Percolating Systems Exhibit Phase Transition (1/2)

Eg. (isotropic Bernoulli percolation)

Thm. Bond percolation on a d-dim lattice $\mathbb{Z}^{d\geq 2}$ has a percolation threshold $0 < p_c^{\text{bond}} < 1$.

Thm. (Harris-Kesten, 2D square bond)

- 1960 (Harris): $p_c \geq 1/2$
- 1980 (Kesten): $p_c \leq 1/2$

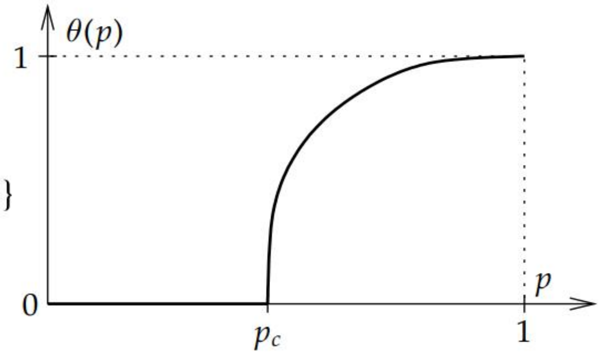
Fact. The observable $\theta(p)$ is zero for $p < p_c$, bounded increasing for $p > p_c$, and continuous at p_c $(d \leq 2, d \geq 19)$.

lattice	p_c (site percolation)	p_c (bond percolation)
honeycomb	0.6962	0.65271*
square	0.592746	0.50000*
triangular	0.50000*	0.34729*

Table. Percolation for $p > p_c$ (percolation threshold).

$$\theta(p) = \mathbb{P}_p(|C| = \infty)$$

$$p_c = \sup\{p : \theta(p) = 0\}$$

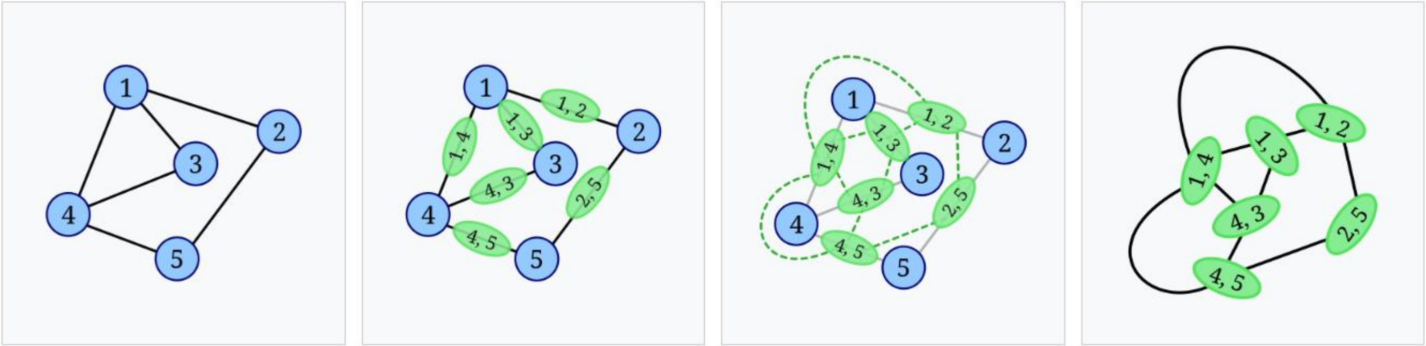


Duminil-Copin, Hugo (2018); Beffara and Sidoravicius (2008)

Percolating Systems Exhibit Phase Transition (2/2)

Exercise. (*Duminil-Copin, Ex. 6*)

Bond percolation on a graph $G(E, V)$ is equivalent to site percolation on its line graph $L(G)$. However, given a graph $\overline{G}(\overline{E}, \overline{V})$, there does not necessarily exists a graph G such that $L(G) = \overline{G}(\overline{E}, \overline{V})$.



Graph *G*

Vertices in L(*G*) constructed from edges in *G*

Added edges in L(*G*)

The line graph L(*G*)

Cor. $p_c^{\text{bond}} \le p_c^{\text{site}} \le 1 - (1 - p_c^{\text{bond}})^d$.

Site percolation is more general than bond percolation

Duminil-Copin, Hugo (2018). *Introduction to Beroulli percolation*

Directed Percolation (DP) and Its Dynamics

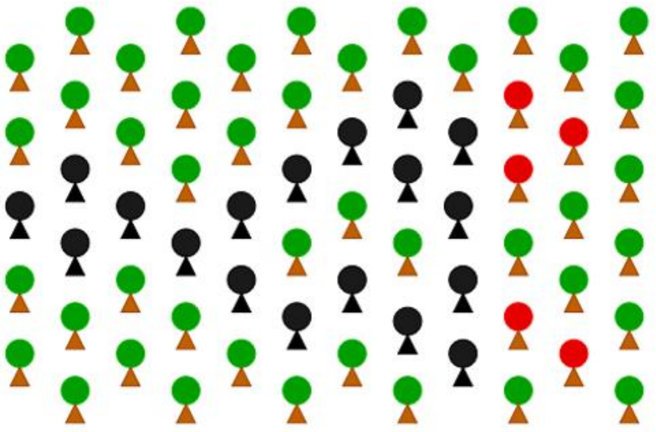
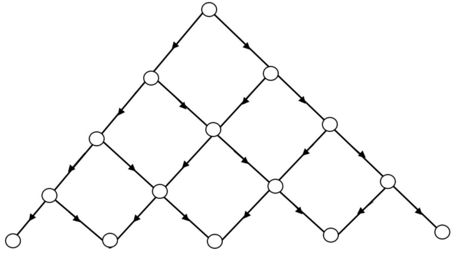


Fig. Forest fire spread from burning trees (red) to healthy trees (green), leaving burnt dead trunk (black).

This is a simple epidemic process (SEP)

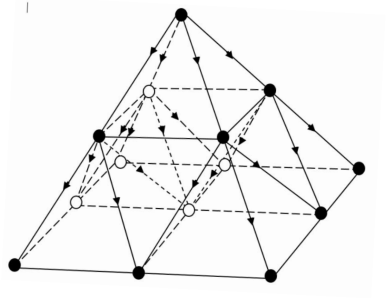
Epidemic model on a lattice will help us build a field theory formulation.

Eg. (directed percolation models)



(1+1)D square lattice

$$p_c^{\mathrm{site}} = 0.7055\ldots$$
$$p_c^{\mathrm{bond}} = 0.6447\ldots$$



(2+1)D BCC lattice

$$p_c^{\mathrm{site}} = 0.3446\ldots$$
$$p_c^{\mathrm{bond}} = 0.2873\ldots$$

Beffara and Sidoravicius (2008)

Universality Classes of Phase Transition

Obs. (universality)
Order params. of classes of systems exhibiting universality are less sensitive to details of the system near critical value. Power-law scaling:

$$a = a_0|\beta - \beta_c|^\alpha.$$

Def. (universality class)
A universality class is an equivalence class of models $[A]$, with relation $A \sim B$ if A, B have the same critical exponents.

Fact. DP defines a universality class DP

Eg. (exponents for 2D percolation)

$$\alpha = -2/3 \quad \beta = 5/36 \quad \gamma = 43/18$$
$$\delta = 91/5 \quad \nu = 4/3 \quad \eta = 5/24$$

Eg. (universality properties)
(1) SEP $\in \mathrm{DP}$
(2) liquid-gas transition $\in$ $\mathrm{3D\ Ising\ model}$

List of known universality classes: https://en.wikipedia.org/wiki/Universality_class

Path Integral Formulation From Reaction-Diffusion System

Construct a field theory for percolation from reaction-diffusion (RD) systems

Eg. (RD model with an active-to-absorbing-state phase transition)

Agent A hops to nearest neighbors on a lattice $\mathbb{Z}^2$

$$|P(t)\rangle = \sum_{\{n,m\}} P(\{n,m\},t)\ |\{n,m\}\rangle$$ bosonic Fock space

$$\partial_t\,|P(t)\rangle = -H\,|P(t)\rangle$$ (classical) master equ.

$A \underset{\kappa}{\overset{\rho}{\rightleftarrows}} 2A\,,$	$A \overset{\sigma}{\to} \varnothing\,,$
birth and propagation	agents may die

Hamiltonian: $H = H_{\text{diff}} + H_{\text{reac}}$

$$H_{\text{diff}} = \lambda \sum_{<i,j>} \big(\hat{a}_i - \hat{a}_j\big) a_i = \frac{\lambda}{2} \sum_{<i,j>} \big(\hat{a}_i - \hat{a}_j\big)\big(a_i - a_j\big)\,,$$

$$H_{\text{reac}} = \sum_i \Big[\rho\big(1-\hat{a}_i\big)\hat{a}_i\,a_i + \kappa\big(\hat{a}_i - 1\big)\hat{a}_i a_i^2 + \sigma\big(\hat{a}_i - 1\big)a_i + \mu\big(\hat{a}_i - \hat{b}_i\big)a_i + \nu\big(\hat{a}_i - 1\big)\hat{b}_i\,b_i\,a_i\Big]\,.$$

Dynamic Response Functional and Path Integral

Construct a field theory for percolation from birth-death SEPs

$$\partial_t\, n(\mathbf{r},t) = \mathcal{V}(\mathbf{r},t)\,, \quad \partial_t\, m(\mathbf{r},t) = \lambda\, n(\mathbf{r},t)\,,$$

The statistical properties of the stochastic process are fully encoded in the simultaneous probability density of the history $\{n(t_0), n(t_1), \ldots, n(t_k), \ldots\}$

$$\overline{\exp\left(\Delta \sum_k \mathcal{V}(t_k)\, \tilde{n}(t_{k+1})\right)} = \exp\left(\Delta \sum_k \sum_{l_k=1}^{\infty} \frac{\tilde{n}(t_{k+1})^{l_k}}{l_k!} K_{l_k}[n(t_k), m(t_k)]\right)$$

$$\mathcal{P}(\{n(t)\}) = \int \mathcal{D}[\tilde{n}]\, \exp \int dt \left\{ \sum_{l=1}^{\infty} \frac{\tilde{n}(t)^l}{l!} K_l[n(t), m(t)] - \tilde{n}(t)\, \partial_t n(t) \right\}.$$

Dynamic response functional: this acts as the effective action

$$\mathcal{J} = \int dt \left\{ \tilde{n}(t)\Big[\partial_t\, n(t) - K_1[n(t), m(t)]\Big] - \frac{1}{2}\, \tilde{n}(t)^2\, K_2[n(t), m(t)] \right\}$$

Rmk. The relevant cumulants K here are the mean and the variance

$$\mathcal{D}[\tilde{n}, n] \;=\; \textstyle\prod_{\mathbf{r},t}\, d\tilde{n}(\mathbf{r},t)\, dn(\mathbf{r},t)/2\pi i.$$

Renormalizatiom Group Equations

Wilsonian renormalization

critical point ⟶ RG flow fixed points

Rmk. The critical dimensions for DP is $d_c^{DP} = 4$

Eg. (general method of RG)
Replace a collection of sites by super-sites and study percolation of the new super-lattice

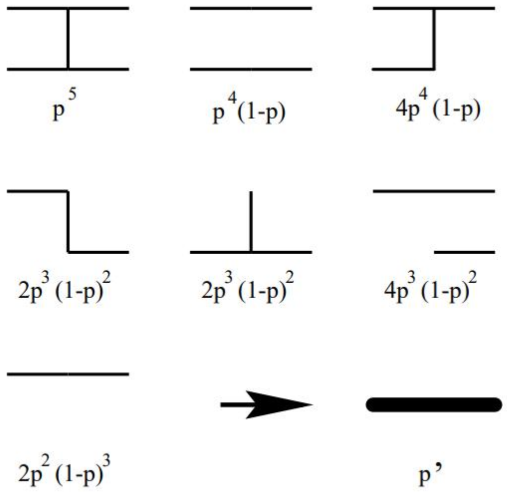


Fig. These combination of bonds lead to a horizontal super-bond after renormalization

Romer, Rudolf A. (2001). *Percolation, Renormalization and the Quantum-Hall Transition*

Renormalized FT for Directed Percolation (DP) (1/2)

$$\mathcal{J}_{\mathrm{DP}} = \int d^d r\, dt \left\{ \tilde{s}\Big[\partial_t + \lambda(\tau - \nabla^2) + \frac{\lambda g}{2}(s - \tilde{s})\Big] s - \lambda h\, \tilde{s} \right\}$$

$$\Downarrow$$

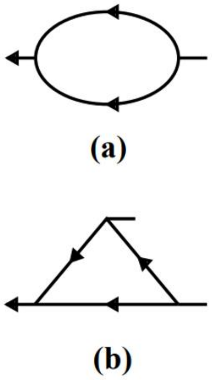
$$\mathring{G}(\mathbf{q}, t) = \theta(t)\, \exp\big[-\mathring{\lambda}(\mathring{\tau} + q^2) t\big]$$

$$\langle s(\mathbf{r}, t)\tilde{s}(\mathbf{r}', t')\rangle_0 =: G(\mathbf{r} - \mathbf{r}', t - t'),$$

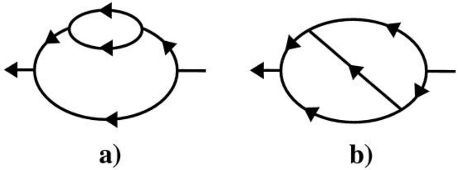
$$G(\mathbf{r}, t) = \int_{\mathbf{q},\omega} \frac{\exp(i\mathbf{q}\cdot\mathbf{r} - i\omega t)}{-i\omega + \lambda(\mathbf{q}^2 + \tau)},$$

propagator

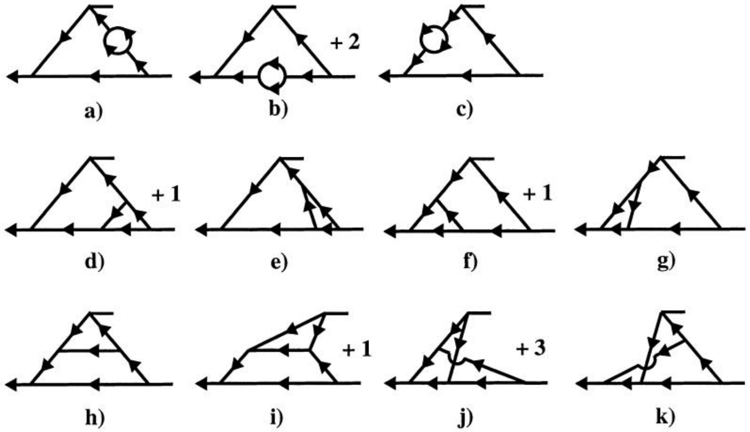
Identify the loop diagrams up to time-ordering:



(1) One-loop diagrams



(2) Two-loop self-energy



(3) Two-loop vertex diagrams with #(time-ordering)

Janssen, Hans-Karl (2000). *Directed Percolation with Colors and Flavors*

Renormalized FT for Directed Percolation (DP) (2/2)

RG functions and IR-stable fixed point

$$\beta(u) = \left[-\varepsilon + \frac{3u}{2} - \left(169 + 106 \ln\frac{4}{3}\right)\frac{u^2}{128} + O(u^3)\right] u \quad \Longrightarrow \quad u_* = \frac{2\varepsilon}{3}\left[1 + \left(\frac{169}{288} + \frac{53}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right]$$

DP has 3 independent critical exponents:

$$\eta = -\frac{\varepsilon}{6}\left[1 + \left(\frac{25}{288} + \frac{161}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right], \qquad \text{correlation-length exponent } \upsilon$$

$$z = 2 - \frac{\varepsilon}{12}\left[1 + \left(\frac{67}{288} + \frac{59}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right], \qquad \text{dynamic exponent } z$$

$$\nu = \frac{1}{2} + \frac{\varepsilon}{16}\left[1 + \left(\frac{107}{288} - \frac{17}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right], \qquad \text{anamolous dimension } \eta$$

$$\beta = \nu\frac{d+\eta}{2} = 1 - \frac{\varepsilon}{6}\left[1 - \left(\frac{11}{288} - \frac{53}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

Rmk. One of the simplest model of a system with a continuous phase transition.

Conclusion

Percolation processes exhibit phase transitions

Near-critical behaviors are described by field theories with few parameters

Path integral with the dynamic response functional as the appropriate effective action. This formalism gives a systematic way to:

- Classify universality classes
- Compute critical exponents and RG functions

References

[1] Hans-Karl Janssen, Uwe C. Tauber (2018). *The field theory approach to percolation*

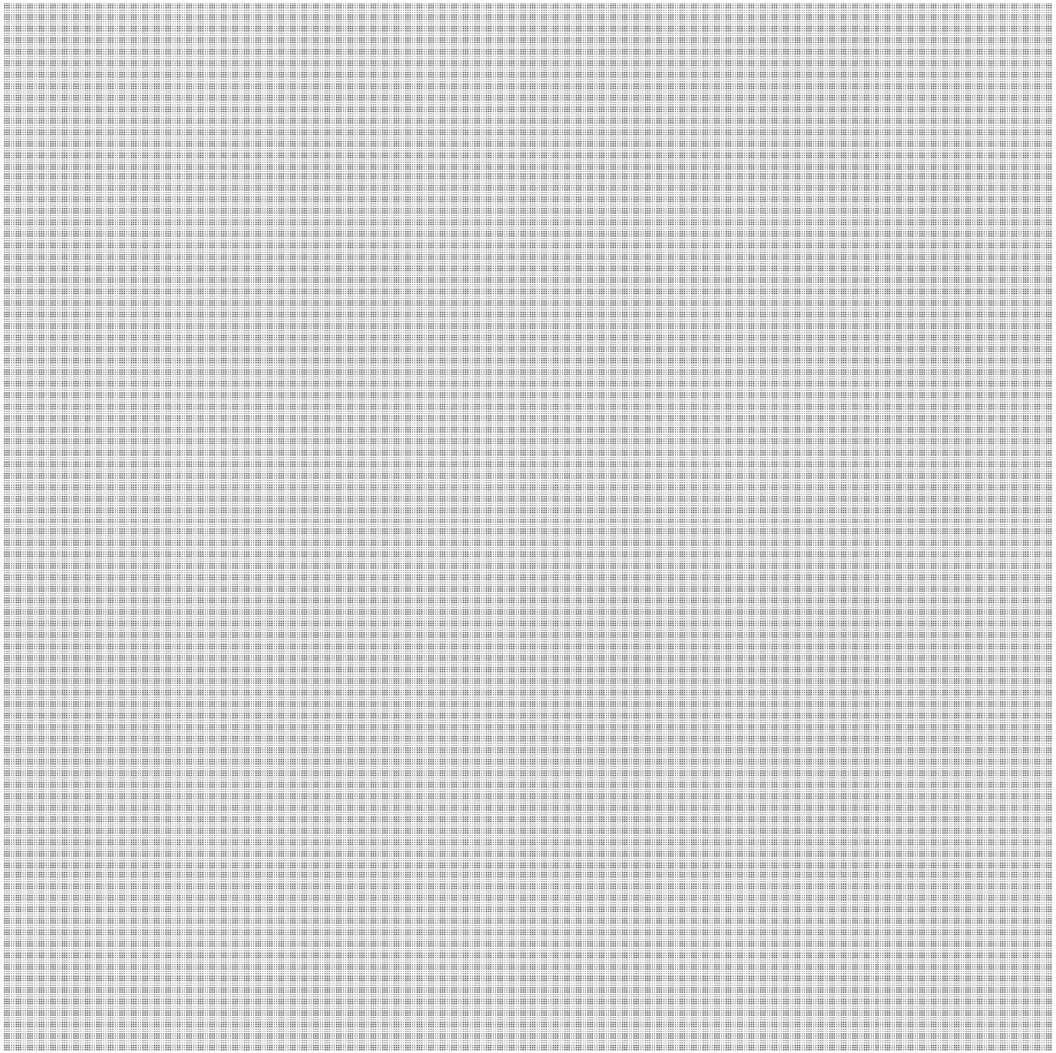
[2] Duminil-Copin, Hugo (2018). *Introduction to Bernoulli percolation*

[3] Beffara and Sidoravicius (2005). *Percolation theory*

[4] Janssen, Hans-Karl (2000). *Directed Percolation with Colors and Flavors*

[5] Romer, Rudolf A. (2001). *Percolation, Renormalization and the Quantum-Hall Transition*

[6] Wikipedia - Universality class https://en.wikipedia.org/wiki/Universality_class



Microscopic Model to Stochastic Equations

In order to compute statistical averages it is necessary to introduce the projection state $\langle\cdot| = \langle 0| \prod_i \exp(a_i + b_i)$. Using the identity $\langle\cdot|\,\hat{a}_i = \langle\cdot| = \langle\cdot|\,\hat{b}_i$ one easily finds the expectation value of an observable $A(\{n,m\})$ at time t:

$$\langle A\rangle(t) = \sum_{\{n,m\}} A(\{n,m\})\, P(\{n,m\},t) = \langle\cdot|A(\{\hat{a}a, \hat{b}b\})|P(t)\rangle\,, \qquad (15)$$

where in the last expression n and m are replaced with the operators $\hat{a}a$ and $\hat{b}b$, respectively. The formal solution of the equation of motion (13) reads $|P(t)\rangle = \exp(-tH)\,|P(0)\rangle$. Following standard procedures [23,25] the expectation value (15) can be expressed as a path integral

$$\langle A\rangle(t) = \int \mathcal{D}[\hat{a},\hat{b},a,b]\; A(\{\hat{a}a,\hat{b}b\})\; \exp(-S[\hat{a},\hat{b},a,b])\,, \qquad (16)$$

with an exponential weight that defines a field theory action. After applying a (naive) continuum limit, the action becomes

$$S = \int d^d r\, dt \Big[(\hat{a}-1)\,\partial_t a + \lambda\,\nabla\hat{a}\cdot\nabla a + (\hat{a}-1)\,(\sigma - \rho\,\hat{a} + \kappa\,\hat{a}a)\,a$$
$$+ (\hat{b}-1)\,\partial_t b + \mu(\hat{a}-\hat{b})\,a + \nu(\hat{a}-1)\,\hat{b}ba\Big]\,. \qquad (17)$$

Renormalization Group Equations

We rename the original bare fields and parameters according to $s \to \mathring{s}$, $\tilde{s} \to \mathring{\tilde{s}}$, $\tau \to \mathring{\tau}$, etc. In accord with the symmetries (26) we choose the following multiplicative renormalizations

$$\mathring{s} = Z^{1/2} s\,, \qquad \mathring{\tilde{s}} = \tilde{Z}^{1/2}\tilde{s}\,, \qquad G_\varepsilon \mathring{g}^2 = \tilde{Z}^{-1} Z_\lambda^{-2} Z_u u \mu^\varepsilon\,, \tag{37a}$$

$$\mathring{\lambda} = (Z\tilde{Z})^{-1/2} Z_\lambda \lambda\,, \qquad \mathring{\tau} = Z_\lambda^{-1} Z_\tau \tau + \mathring{\tau}_c\,, \qquad \mathring{h} = Z^{1/2} Z_\lambda^{-1} h\,, \tag{37b}$$

$$\tilde{Z} = Z \quad \text{for DP}\,, \qquad \tilde{Z} = Z_\lambda \quad \text{for dIP}\,, \tag{37c}$$

where $G_\varepsilon = \Gamma(1+\varepsilon/2)/(4\pi)^{d/2}$ is a convenient amplitude. u represents the dimensionless coupling constant, and $\tau = 0$ at the critical point. The renormalization constants $Z_{\dots} = Z_{\dots}(u, \mu/\Lambda, \varepsilon)$ can be chosen in a UV-renormalizable theory in such a way that

Scaling Form and Critical Exponents

Define the RG functions:

$$\beta(u) = \left.\frac{\partial u}{\partial \ln \mu}\right|_0, \qquad \gamma(u) = \left.\frac{\partial \ln Z}{\partial \ln \mu}\right|_0, \qquad \tilde{\gamma}(u) = \left.\frac{\partial \ln \tilde{Z}}{\partial \ln \mu}\right|_0,$$

$$\kappa(u) = \left.\frac{\partial \ln \tau}{\partial \ln \mu}\right|_0, \qquad \zeta(u) = \left.\frac{\partial \ln \lambda}{\partial \ln \mu}\right|_0.$$

$$\left[\mu\frac{\partial}{\partial\mu} + \zeta\lambda\frac{\partial}{\partial\lambda} + \kappa\tau\frac{\partial}{\partial\tau} + \beta\frac{\partial}{\partial u} + \frac{1}{2}(N\gamma + \tilde{N}\tilde{\gamma})\right] G_{N,\tilde{N}}(\{\mathbf{r},t\},\tau,u,\lambda,\mu) = 0$$

Callan–Symanzik equ. describes the evolution of the *n*-point corr. function

Fact. Percolation processes near the critical point are asymptotically described by universal scaling functions with 3 scaling exponents and 4 non-universal amplitudes

Renormalized FT for Directed Percolation (DP) (2/4)

$$\mathcal{J}_{\mathrm{DP}} = \int d^d r\, dt \left\{ \tilde{s}\left[\partial_t + \lambda(\tau - \nabla^2) + \frac{\lambda g}{2}(s - \tilde{s})\right] s - \lambda h\, \tilde{s} \right\}$$

$$\Downarrow$$

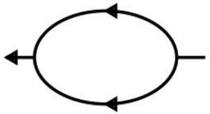
$$\mathring{G}(\mathbf{q}, t) = \theta(t)\, \exp\left[-\mathring{\lambda}(\mathring{\tau} + q^2)t\right]$$

$$\langle s(\mathbf{r}, t)\tilde{s}(\mathbf{r}', t')\rangle_0 =: G(\mathbf{r} - \mathbf{r}', t - t'),$$

$$G(\mathbf{r}, t) = \int_{\mathbf{q},\omega} \frac{\exp(i\mathbf{q}\cdot\mathbf{r} - i\omega t)}{-i\omega + \lambda(\mathbf{q}^2 + \tau)},$$

propagator

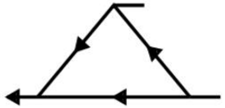
One-loop diagrams for DP



(a)

$$= -\frac{\mathring{\lambda}\mathring{g}^2}{2}\int_{\mathbf{p}} \frac{1}{i\omega/\mathring{\lambda} + 2\mathring{\tau} + (\mathbf{p} - \mathbf{q}/2)^2 + (\mathbf{p} + \mathbf{q}/2)^2}$$

$$= \frac{G_\varepsilon}{2\varepsilon}\, \mathring{\tau}^{-\varepsilon/2}\, \mathring{\lambda}\mathring{g}^2 \left(\frac{2\mathring{\tau}}{2 - \varepsilon} + \frac{i\omega}{2\mathring{\lambda}} + \frac{\mathbf{q}^2}{4}\right) + \dots .$$



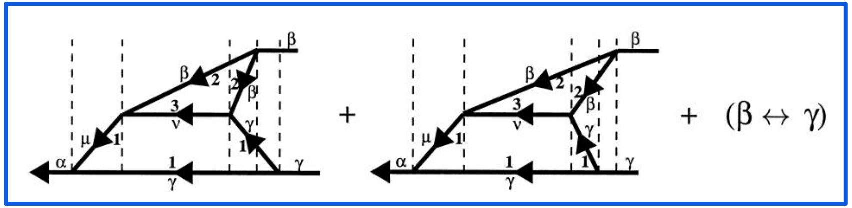
(b)

$$= 2\mathring{\lambda}\mathring{g}^3 \int_{\mathbf{p}} \frac{1}{\left[2(\mathring{\tau} + \mathbf{p}^2)\right]^2} = \frac{G_\varepsilon}{\varepsilon}\, \mathring{\tau}^{-\varepsilon/2}\, \mathring{\lambda}\mathring{g}^3$$

Renormalized FT for Directed Percolation (DP) (SI)

2. Two-loop vertex diagrams for DP

Eg. Calculation details (i):



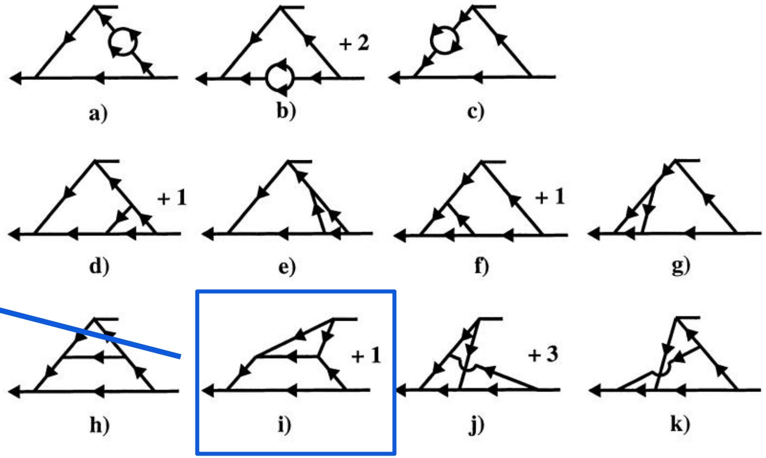
External frequencies and momenta can be set to zero

$$5(i) = -\frac{\lambda^5}{8}\sum_{\mu,\nu}(\delta_{\alpha\mu}g_{\alpha\nu}+\delta_{\alpha\gamma}g_{\alpha\mu})(\delta_{\mu\nu}g_{\mu\beta}+\delta_{\mu\beta}g_{\mu\nu})(\delta_{\alpha\mu}g_{\alpha\nu}+\delta_{\alpha\gamma}g_{\alpha\mu})\, g_\beta g_\gamma$$

$$\times \int_{q_1,q_2}\left(\frac{1}{(2\kappa_1)(\kappa_1+\kappa_2+\kappa_3)(2\kappa_1+2\kappa_2)(2\kappa_1)} + \frac{1}{(2\kappa_1)(\kappa_1+\kappa_2+\kappa_3)(2\kappa_1+2\kappa_2)(2\kappa_2)}\right)+(\beta\leftrightarrow\gamma)$$

$$= -\frac{\lambda g^2}{32}G_\varepsilon^2\tau^{-\varepsilon}(\delta_{\alpha\beta}g_{\alpha\gamma}+\delta_{\alpha\gamma}g_{\alpha\beta})(g(g_{\beta\gamma}+g_{\gamma\beta})+g_{\alpha\beta}g_{\alpha\gamma}+g_{\beta\gamma}g_{\gamma\beta})\, I_{12,1}$$

$\Longrightarrow$



Two-loop vertex diagrams with #(time-ordering)

$$\Gamma^{5(i)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16}G_\varepsilon^2\tau^{-\varepsilon}g_{\alpha\beta}(g(g_{\beta\alpha}+g_{\alpha\beta})+g_{\alpha\beta}g+g_{\beta\alpha}g_{\alpha\beta})\, I_{12,1}$$

$$\kappa_i = \lambda(\tau+q_i^2),\ \mathbf{q}_3 = \mathbf{q}_1+\mathbf{q}_2$$

repeat for (a) ~ (k) !

Janssen, Hans-Karl (2000). *Directed Percolation with Colors and Flavors*

Renormalized FT for Directed Percolation (DP) (1/3)

One of the simplest model of a nonequilibrium system with a continuous phase transition.

Rmk. DP is the non-equilibrium analog of φ^4 theory for Ising universality.

$$\mathring{s} = Z^{1/2} s\,, \qquad \mathring{\tilde{s}} = \tilde{Z}^{1/2}\tilde{s}\,, \qquad G_\varepsilon \mathring{g}^2 = \tilde{Z}^{-1} Z_\lambda^{-2} Z_u u \mu^\varepsilon\,, \tag{37a}$$

$$\mathring{\lambda} = (Z\tilde{Z})^{-1/2} Z_\lambda \lambda\,, \qquad \mathring{\tau} = Z_\lambda^{-1} Z_\tau \tau + \mathring{\tau}_c\,, \qquad \mathring{h} = Z^{1/2} Z_\lambda^{-1} h\,, \tag{37b}$$

$$\tilde{Z} = Z \quad \text{for DP}\,, \qquad \tilde{Z} = Z_\lambda \quad \text{for dIP}\,, \tag{37c}$$

Field theory construction: find cumulants K for the dynamics response functions

$$\mathcal{J} = \int dt \left\{ \tilde{n}(t) \Big[\partial_t n(t) - K_1[n(t), m(t)] \Big] - \frac{1}{2} \tilde{n}(t)^2 K_2[n(t), m(t)] \right\}$$

All Diagrams

$$I_{kl;m} = G_\varepsilon^{-2}\tau^\varepsilon \int_{\mathbf{q}_1,\mathbf{q}_2} \frac{1}{(q_1^2+\tau)^k (q_2^2+\tau)^l (q_1^2+q_2^2+(q_1+q_2)^2+3\tau)^m}$$

where $G_\varepsilon = \Gamma(1+\varepsilon/2)/(4\pi)^{d/2}$, $\varepsilon = 4-d$, and $\int_{\mathbf{q}} \cdots = (2\pi)^{-d}\int d^d q \ldots$

$$\Gamma^{5(a)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g(2g+g_{\alpha\beta}+g_{\beta\alpha})\, I_{30,1}$$

$$\Gamma^{5(b)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g(2g+g_{\alpha\beta}+g_{\beta\alpha})(I_{20,2}+I_{30,1})$$

$$\Gamma^{5(c)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g(2g+g_{\alpha\beta}+g_{\beta\alpha})\, I_{30,1}$$

$$\Gamma^{5(d)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g(2g+g_{\alpha\beta}+g_{\beta\alpha})(2I_{11,2}+I_{12,1})$$

$$\Gamma^{5(e)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g(2g+g_{\alpha\beta}+g_{\beta\alpha})\, I_{12,1}$$

$$\Gamma^{5(f)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}((g_{\alpha\beta}^2+g_{\beta\alpha}^2)+2gg_{\beta\alpha}+4g^2)(2I_{11,2}+I_{12,1})$$

$$\Gamma^{5(g)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}(g(2g+g_{\alpha\beta})+g_{\alpha\beta}g_{\beta\alpha})\, I_{12,1}$$

$$\Gamma^{5(h)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}(g_{\alpha\beta}^2+g_{\beta\alpha}^2+2gg_{\beta\alpha}+4g^2)\, I_{20,2}$$

$$\Gamma^{5(j)}_{\alpha,\alpha\beta} = \frac{3\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}^2(g+g_{\beta\alpha})\, I_{11,2}$$

$$\Gamma^{5(k)}_{\alpha,\alpha\beta} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}(g_{\alpha\beta}^2+g_{\beta\alpha}^2+2gg_{\beta\alpha}+4g^2)\, I_{11,2}$$

Renormalized FT for Directed Percolation (DP) (4/4)

RG functions and IR-stable fixed point are determined

$$Z = 1 + \frac{u}{4\varepsilon} + \left(\frac{7}{\varepsilon} - 3 + \frac{9}{2}\ln\frac{4}{3}\right)\frac{u^2}{32\varepsilon} + O(u^3)\,,$$

$$Z_\lambda = 1 + \frac{u}{8\varepsilon} + \left(\frac{13}{4\varepsilon} - \frac{31}{16} + \frac{35}{8}\ln\frac{4}{3}\right)\frac{u^2}{32\varepsilon} + O(u^3)\,,$$

$$Z_\tau = 1 + \frac{u}{2\varepsilon} + \left(\frac{16}{\varepsilon} - 5\right)\frac{u^2}{32\varepsilon} + O(u^3)\,,$$

$$Z_u = 1 + \frac{2u}{\varepsilon} + \left(\frac{4}{\varepsilon} - 1\right)\frac{7u^2}{8\varepsilon} + O(u^3)\,.$$

$\Longrightarrow$

$$\beta(u) = \left[-\varepsilon + \frac{3u}{2} - \left(169 + 106\ln\frac{4}{3}\right)\frac{u^2}{128} + O(u^3)\right]u$$

$$u_* = \frac{2\varepsilon}{3}\left[1 + \left(\frac{169}{288} + \frac{53}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right]$$

DP critical exponents:

$$\eta = \gamma(u_*)\,, \qquad \tilde{\eta} = \tilde{\gamma}(u_*)\,,$$

$$\nu^{-1} = 2 - \kappa(u_*)\,, \qquad z = 2 + \zeta(u_*)$$

with $\delta_{N,\tilde{N}} = (N + \tilde{N})\beta/\nu$.

$$\eta = -\frac{\varepsilon}{6}\left[1 + \left(\frac{25}{288} + \frac{161}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

$$z = 2 - \frac{\varepsilon}{12}\left[1 + \left(\frac{67}{288} + \frac{59}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

$$\nu = \frac{1}{2} + \frac{\varepsilon}{16}\left[1 + \left(\frac{107}{288} - \frac{17}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

$$\beta = \nu\frac{d+\eta}{2} = 1 - \frac{\varepsilon}{6}\left[1 - \left(\frac{11}{288} - \frac{53}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$