A model of gravitational lensing using an optical-mechanical analogy

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Abstract

Gravitational lensing is a strong evidence supporting the theory of General Relativity. Despite its complexity, we used an optical analogy of spatially varying refractive index to mimic the effect of gravitational lens: the refractive index n(r) is unity at large distances and increases near the mass, mimicking how a gravitational field slows light. Using Fermat's principle of least time, we obtain the Euler–Lagrange equation that governs ray trajectories. This equation yields a conserved quantity (flight invariant) along each ray's path, analogous to the impact parameter of a light ray in a gravitational field. By solving the Euler–Lagrange equation, we can obtain the trajectory of each light ray through the medium. Upon finding the apparent position of the source(such as a star) through these trajectories, we then rotate these ray paths based on the spherical symmetry of the model, so as to simulate the image of a disk. The resulting images exhibit characteristic ring-like and multiple-image patterns just like those seen in real observations, suggesting that the refractive-index analogy accurately reproduces known gravitational lensing phenomena, providing an intuitive and accessible framework for physics students.

Contents

I. INTRODUCTION

The theory of relativity comprises two interrelated frameworks introduced by Albert Einstein. Special Relativity (1905) governs the behavior of physical phenomena in the absence of gravity, unifying space and time and predicting effects such as time dilation and length contraction. General Relativity (1915) extends this framework by describing gravity as the curvature of spacetime produced by mass. Together, these theories revolutionized twentieth-century physics by superseding the Newtonian paradigm and introducing new concepts such as four-dimensional spacetime and relativistic time dilation. In particular, Einstein's theory transformed our understanding of gravitation, leading to predictions of novel phenomena (e.g. black holes, gravitational waves) and reshaping cosmology and astrophysics.

One notable prediction of general relativity is gravitational lensing – the deflection of light by massive objects. When a massive body (such as a galaxy or cluster) significantly curves spacetime, it causes the path of a light ray passing nearby to bend, much like an optical lens. As a result, light from a distant source can be distorted, magnified, or split into multiple images. Gravitational lensing thus provides a dramatic and observable demonstration of Einstein's theory, and it has become a powerful tool for studying objects that would otherwise be too faint or distant to see.

This report focuses on modeling gravitational lensing using an optical-mechanical analogy. We adopt isotropic coordinates to express a spherically symmetric spacetime metric in a form that is conformal to flat space. In these coordinates, the gravitational field of a mass can be mapped to an effective refractive index for light(de Felice,F.,1971). Under this analogy, light rays follow trajectories as if traveling through a medium with spatially varying refractive index, enabling the use of geometric optics methods to derive light paths and lens equations. This approach provides an intuitive framework for analyzing lensing geometries while remaining equivalent to solving the null geodesic equations in general relativity.

For the following contents, in section 2, we review the historical context of relativity, including Einstein's original formulation of Special and General Relativity and early empirical tests (such as Eddington's solar eclipse experiment in 1919). In section 3, we introduce the modeling framework for gravitational lensing, explaining the use of isotropic coordinates and the optical analogy (treating gravity as a refractive medium) and in section 5, the detailed derivation of light trajectories and lensing equations within the chosen framework is provided, using analogies to classical optics.

II. HISTORIC NOTES

To understand the theory of relativity and its implications in current scientific developments, in particular the ideas used in the paper we studied, it is helpful to get a basic understanding of the history of its development, and put ourselves in the shoes of the scientific giants that shone along the way. Here we present a historic recap, and theory will follow afterward.

A. The Beginning

The history of the study of light and its interaction with gravitating bodies started with Sir Isaac Newton (1643 – 1727) as early as in the 18th century. In his famous magnum opus, *Opticks* (1704) [2], he proposed the **corpuscular theory of light**, in which he postulated light to be made of small, discrete particles possessing definite momentum and finite velocity ("corpuscules"). This theory, though long known to be wrong from light speed measurement experiments in a medium, such as the Fizaeu experiment (conducted by Hippolyte Fizeau in 1851), paved the way for future progress on the refraction of light. Newton speculated that gravity could act on light particles, potentially bending their paths near massive bodies, though he never formulated it precisely. We shall return to this old theory in a moment for a discussion on light trajectory. In the mean time, well-established scientists like Henry Cavendish (1731 – 1810), the man who "weighed the Earth", also proposed the idea of light bending near massive bodies.

Since the time of Galileo Galilei (1564 – 1642), most scientists have accepted and furthermore adopted the idea that acceleration due to gravity is independent of the object's mass. Hence, it was not a stretch that Cavendish posited that gravity bends light. A long time later in 1921, the Astronomer Royal, Frank Watson Dyson (1868 – 1939), who will appear later as an important part, made excerpts of four of Cavendish's astronomical notes and published them. Among them was "an isolated scrap on the bending of a ray of light by gravitation..." [18]

Already, we see that the effect of light bending under the influence of gravity is not so

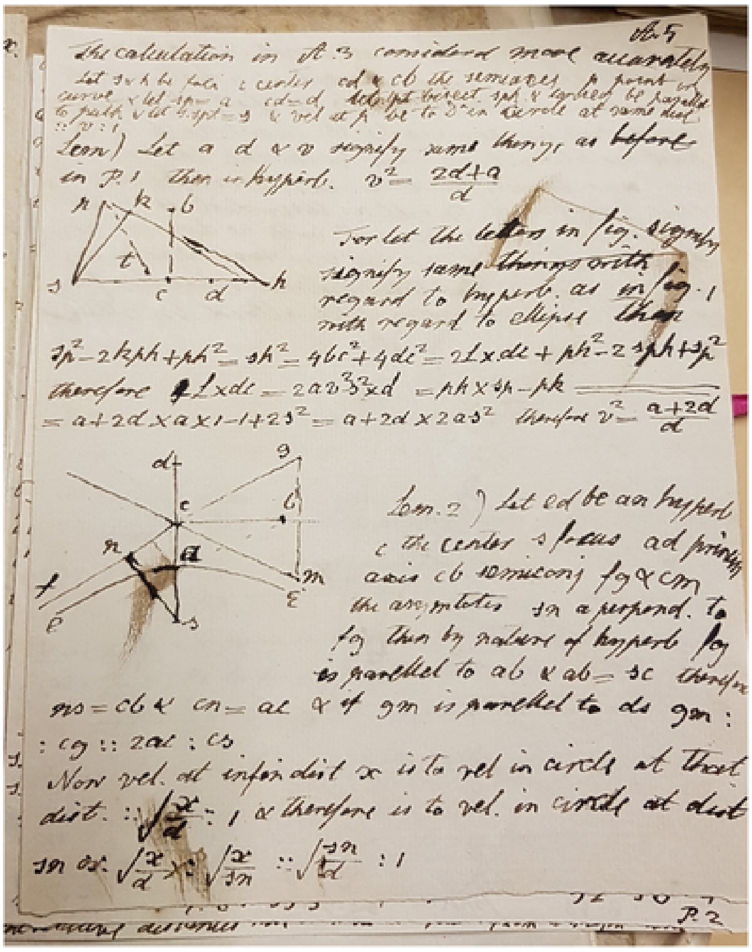


FIG. 1: Cavendish's page "A.5". Here he calculated the velocity that a particle has at an arbitrary point of a hyperbola. For more details and a recap of his original derivation attempt, the reader is welcome to read the source paper [18].

much a revolutionary insight brought forth by Einstein alone in the onset of the twentieth century. Instead, it is appropriate to say that he stood on the shoulders of many giants in the process of formulating his general theory of relativity.

However, it wasn't until 1801 that the said deflection was first explicitly calculated and published. The man in question was Johann Georg von Soldner (1776 - 1833), a less well known physicist of the time, who calculated the deflection of light by gravity using Newtonian mechanics. He predicted a small deflection of around 0.87 arcseconds for light grazing the Sun. Unfortunately, since the deviation was way too tiny and there was no way to make extensive observations during the day, not to say at the direction of the sun, his calculation remained a theoretical curiosity and soon fell into oblivion.

B. Theory of Relativity and the Eddington Expedition

In 1911, Albert Einstein (1879 – 1955), not knowing that an identical result had been derived more than 100 years ago, revisited the idea of light bending using an early version of special relativity. He arrived at a similar value as Soldner, and the publication brought much spotlight for the problem. In modern notation, the deflection angle is given by

$$\delta = \sqrt{\frac{GM}{bc^2}}, \tag{1}$$

where b is the impact parameter.

In 1912, when the general theory of relativity had not yet been proposed, Einstein predicted the optical effect of what is now known as "Einstein rings" based on calculations done solely using special relativity.

The value for deflection angle turns out to be wrong, however, in the context of general relativity. After adventing the general theory of relativity in 1915, Einstein recalculated the correct value of deflection to be

$$\delta = \sqrt{\frac{4GM}{bc^2}}, \tag{2}$$

twice the original value. Now the question of which value was correct was in urgent need to be tested, since its measurement implied a test of whether Einstein's relativity would triumph over the Newtonian theory of mechanics.

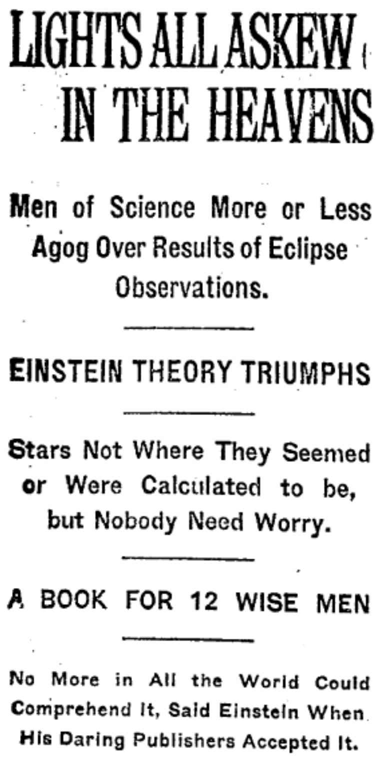
The sun was naturally the best subject of experiment given the measurement abilities of the time, but the problem of how to measure directly at the direction of the sun and still be able to pick out the light from distant stars remained a problem. The measurement of deflection of light around the sun would comprise one of the three classic tests of general relativity: light deflection around the sun, perihelion precession of mercury, and gravitational redshift. [3]

Fortunately, the first observational confirmation came in 1919. During the solar eclipse of 1919, led by Sir Arthur Eddington (1882 – 1944), joined by his colleague Sir Frank Watson Dyson, whom we have mentioned earlier, a group of astronomers confirmed Einstein's prediction by observing distant stars whose light passed near the sun and was deflected. [8] This provided the first empirical evidence for General Relativity — a landmark moment in physics. The publication made headlines in the newspaper at the time, and Einstein - and his theory of relativity - became known around the world. Despite it tremendous success, Eddington's expedition wasn't without turmoil.

The idea of an expedition was conceived as early as in 1916, just one year after Einstein's theory of general relativity was published. Dyson chose the 1919 solar eclipse because it would take place with the Sun in front of a bright group of stars called the Hyades, and the brightness would enhance the quality of observation. When 1919 came, two teams of two people each were to be sent to make observations of the eclipse at different locations: the West African island of Príncipe, led by Eddington and joined by his colleague Edwin Turner Cottingham at the Cambridge Observatory, and the expedition to the Brazilian town of Sobral, carried out by Andrew Crommelin and Charles Rundle Davidson from the Greenwich Observatory in London.

Despite the grand success, doubts soon surfaced in the scientific world about whether Eddington's results were sufficiently accurate and without biases, to be used as concrete evidence of general relativity. Eclipse measurements using visible light retained considerable uncertainty, and it was only radio-astronomical measurements in the late 1960s that definitively verified the amount of deflection as being in complete agreement with the value predicted by general relativity.

Eclipse measurements of this kind, using visible light, retained considerable uncertainty, and it was only radio-astronomical measurements in the late 1960s that definitively showed that the amount of deflection was the full value predicted by general relativity, and not half that number as predicted by a "Newtonian" calculation.


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

FIG. 2: The New York Times reported on the Eddington experiment and Einstein's theory on November 10, 1919.

Interestingly, the 1919 results were used as part of the systematic efforts by Nobel laureate Philipp Lenard to discredit Einstein. This was in a time when Germany was overcast by radical national socialist views, and German scientists regarded "German physics" as superior over "Jewish physics". As mentioned above, Johann Georg von Soldner had, unbeknownst to Einstein, derived from Newtonian gravity for starlight bending around a massive object, which corresponds exactly to Einstein's own erroneous derivation of 1911. Lenard claimed that this proved Einstein to be a plagiarist, and attacked Einstein for this.

What proved troublesome for Einstein wasn't just his Jewish identity, but also the fact that he resided in Germany at the time, and many outside Germany considered him to be *German*. After the widespread success of Einstein's theory was passed on around the world, and various headlines were made about it, it seemed that newspapers in Belgium generally ignored this grand event. Remember that the War had just ended, Belgium mostly overrun by German forces, and so it is suggested that *"the hostility towards Germany among the larger Belgian public, made the country largely inimical to the ideas of a scientist who remained, after all, German"*.[7]

The results from the Sobral expedition in Brazil were discarded due to a defect in the telescopes used, and the discarded results actually fit the Newtonian model better [19]. It is no wonder that some scientists accussed Eddington of making a biased result, in favor of general relativity. Eddington developed the photographs on Príncipe, and attempted to measure the change in the stellar positions during the eclipse. On 3 June, despite the clouds that had reduced the quality of the plates, Eddington wrote: "... one plate I measured gave a result agreeing with Einstein." And it was this sentence that marked the beginning of a new era for physics.

As history tells, Einstein faced backlash from the scientific community, for many at the time could not understand his theory, nor did they believe it to be true. After the expedition, Eddington played a major role in advocating Einstein's theory, now that they had concrete evidence of its validity over the Newtonian model.

After the expedition came to an end, their results were announced at a Royal Society meeting. Later, during a dinner held at the Society, Eddington addressed the participants by reciting a parody verse he penned himself [8]:

"Oh leave the Wise our measures to collate
One thing at least is certain, light has weight
One thing is certain and the rest debate
Light rays, when near the Sun, do not go straight."

This poem was written in the prose style of the *Rubaiyat of Omar Khayyam*, a famous English translation of a Persian text attributed to Omar Khayyam [9].

C. Gravitational Lensing and A New Age for Cosmology

Although being the first to make quantitative prediction of gravitational lenses, Einstein showed much pessimism toward their discovery. He said the following in a 1936 paper: "Of course, there is no hope of observing this phenomenon directly." [15]

In 1937, another prominent name in astronomical physics joined the game. This was Fritz Zwicky (1898 – 1974), a Swiss astronomer from CalTech famous for proposing the existence of dark matter, and for his various works related to astronomy. Before making his name as a prominent astronomer and phsicist, he studied and contributed to the study of ionic crystals and electrolytes, before using the virial theorem to posit the existence of dark matter, calling it *dunkle Materie* [10].

Then, in a 1937 paper, Zwicky posited that galaxies could act as gravitational lenses, following the theory of Einstein [11]. It was not the first time Zwicky made curious claims that seemed too wild to be true, such as being the first to produce artificial meteors and thus launching the first objects into solar orbit [13], and suggesting launching pellets into the sun to cause asymmetrical fusion explosions, thus propelling the entire solar system like a spaceship toward another galaxy [14]. He remarked that with successful implementation, it would only take 2500 years to reach Alpha Centauri, the nearest star (in fact, it is a three-star system) to our home [1].

As remarked by Stephen M. Maurer [12], "When researchers talk about neutron stars, dark matter, and gravitational lenses, they all start the same way: "Zwicky noticed this problem in the 1930s. Back then, nobody listened..."

This time, it turned out that his idea was true, but took just a bit too long for scientists to verify. It wasn't until 1979, 5 years after Zwicky passes away, that a group of English and American scientists, led by Dennis Walsh, Robert Carswell, and Ray Weyman, observed the gravitational lensing effect for the first time.

The discovery was made using the 2.1m Telescope at Kitt Peak National Observatory in

[1] This is said in Wikipedia - Frityz Zwicky to be in the reference [16], but it is in German so I am not fully sure about its correctness

Arizona, and was named the Twin Quasar since it is a quasar that appears as two images due to gravitational lensing. The team noted that two quasars, as shown in the picture below, seemed to be unusually close to each other, and that their visible light spectrum were very close to each other. Although the authors suggested the possibility of a gravitational lens, it was only a tentative result which was subsequently confirmed by various experiments and analyses [17]. Thus, Einstein's previous pessimism that gravitational lenses would never be observed was, very much merrily, disproved.

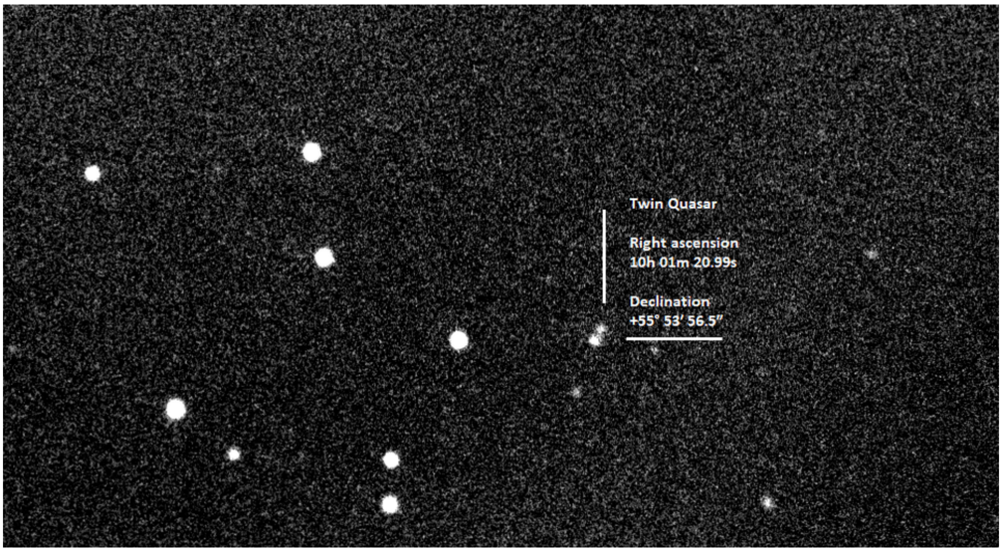


FIG. 3: First image of the Twin Quasar taken by Dennis Walsh et al. at the Kitt Peak National Observatory in Arizona, United States.

Following the discovery of the first gravitational lens, other projects continued searching for and finding more gravitational lensing systems in the sky. These include the more recent Hubble Telescope and James Webb Space Telescope, providing more and more exciting news for the study of astronomy and cosmology.

III. MODEL

A. General Relativity

We know that the metric of a spherically symmetric object with mass m is given by the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{m}{2r}\right)\mathrm{d}t^2 - \left(1 - \frac{m}{2r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2 \tag{3}$$

We use the isotropic coordinate, whose defining characteristic is that its radial coordinate (which is different from the radial coordinate of a Schwarzschild coordinate) is defined so that light cones appear round. However, unlike the Schwarzschild coordinate, the isotropic chart is not well suited for constructing embedding diagrams of these hyperslices. We then use a suitable transformation to obtain the isotropic coordinate. Let $r = r'(1 + \frac{m}{2r'})^2$, where r' is the radius of the isotropic coordinate.

$$1 - \frac{2m}{r} = 1 - \frac{2m}{r'\left(1 + \frac{m}{2r}\right)^2} = \frac{\left(1 + \frac{m}{2r}\right)^2 - \frac{2m}{r'}}{\left(1 + \frac{m}{2r}\right)^2} = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} \tag{4}$$

$$\begin{aligned}\mathrm{d}r &= \mathrm{d}r'\left(1 + \frac{m}{2r}\right)^2 + r'\mathrm{d}\left(1 + \frac{m}{2r}\right)^2 \\ &= \left[\left(1 + \frac{m}{2r}\right)^2 + 2r'\left(1 + \frac{m}{2r}\right)\left(-\frac{m}{2r'^2}\right)\right]\mathrm{d}r' \\ &= \left(1 + \frac{m}{2r'}\right)\left(1 - \frac{m}{2r'}\right)\mathrm{d}r'\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}\mathrm{d}s^2 &= \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2}\mathrm{d}t^2 - \frac{\left(1 + \frac{m}{2r}\right)^2}{\left(1 - \frac{m}{2r}\right)^2}\left(1 + \frac{m}{2r}\right)^2\left(1 - \frac{m}{2r}\right)^2\mathrm{d}r'^2 - r'^2\left(1 + \frac{m}{2r}\right)^4\left(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2\right) \\ &= \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2}\mathrm{d}t^2 - \left(1 + \frac{m}{2r}\right)^4\left(\mathrm{d}r'^2 + r'^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)\end{aligned} \tag{6}$$

The above form of metric, so-called the isotropic coordinate, has a conformal Euclidean spatial part. Therefore, the isotropic coordinate speed of the light $c(r)$ in analogy to classical optics can be obtained through letting $\mathrm{d}s^2 = 0$.

$$0 = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^6}\mathrm{d}t^2 - |\mathrm{d}\mathbf{r}'|^2 \tag{7}$$

$$c^2(r') = \left|\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right|^2 = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^6} \tag{8}$$

In analogy to classical optics, we can read the refraction index (write r' as r)

$$n^2(r) = \frac{\left(1 + \frac{m}{2r}\right)^6}{\left(1 - \frac{m}{2r}\right)^2} \tag{9}$$

B. Simplified refraction index with similar behavior around the event horizon

Using the refractive index derived in previous section, if we introduce a new variable $\rho = r - m/2$ then as $r \to m/2$, Eq (7) becomes

$$n(\rho)^2 = \frac{16(m+\rho)^6}{\rho^2(m+2\rho)^4} \tag{10}$$

which diverges as $1/\rho^2$ in the vicinity of the surface of the event horizon. We now consider an index of refraction with similar behavior in the proximity of the event horizon.

$$n(r)^2 = 1 + \frac{C^2}{r^2}, \quad C = 4m \tag{11}$$

where C is a constant of unit length. Then $r \to 0$ captures the leading-order behavior of light around the event horizon, and $r \to \infty$ captures the motion of light in a flat space-time.

C. General solution to the ray trajectory

To get the trajectory of light, we use the **Fermat's principle** from classical optics in 2D polar coordinate. Using the index of refraction derived from above, we need to maximize the optical path

$$\delta L = \delta \int F(\theta, r(\theta), \dot{r}(\theta)) \mathrm{d}\theta = 0 \tag{12}$$

where $\dot{r}(\theta) = \mathrm{d}r/\mathrm{d}\theta$, and

$$F = n(r)\sqrt{\dot{r}^2 + r^2} \tag{13}$$

The trajectory is given by the Euler-Lagrange equation

$$\frac{\partial F}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial F}{\partial \dot{r}} = 0 \tag{14}$$

$$\frac{\partial F}{\partial r} = n'(r)\sqrt{\dot{r}^2 + r^2} + \frac{n(r)r}{\sqrt{\dot{r}^2 + r^2}} \tag{15}$$

$$\frac{\partial F}{\partial \dot{r}} = \frac{n(r)\dot{r}}{\sqrt{\dot{r}^2 + r^2}} \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial F}{\partial \dot{r}} = \frac{n(r)\ddot{r}}{\sqrt{\dot{r}^2 + r^2}} - \frac{\dot{r}^2 n(r)(\ddot{r} + r)}{(\dot{r}^2 + r^2)^{3/2}} + \frac{n'(r)\dot{r}^2}{\sqrt{\dot{r}^2 + r^2}} = \frac{n(r)(\ddot{r}r^2 - r\dot{r}^2)}{(\dot{r}^2 + r^2)^{3/2}} + \frac{n'(r)\dot{r}^2}{\sqrt{\dot{r}^2 + r^2}} \tag{17}$$

where $n'(r) = \mathrm{d}n/\mathrm{d}r$ is the derivative with respect to r. The Euler-Lagrange equation then becomes

$$n'(r)r^2 + n(r)r = \frac{n(r)(\ddot{r}r^2 - r\dot{r}^2)}{(\dot{r}^2 + r^2)}$$

which then leads to

$$\frac{n'(r)}{n(r)} + \frac{r^2 + 2\dot{r}^2 - \ddot{r}r^2}{\dot{r}^2 r + r^3} = 0 \tag{18}$$

If we define a new variable $u(r) = \dot{r}/r$ and substitute $\dot{r} = ur$ and $\ddot{r} = u\dot{r} + \dot{u}r = u^2 r + u' r\dot{r}$, then the trajectory equation becomes

$$\frac{n'(r)}{n(r)} + \frac{1}{r} - \frac{uu'}{1 + u^2} = 0 \tag{19}$$

Integrating over the radius coordinate r, we obtain

$$rn\left(r\right)\left(1 + u\left(r\right)^2\right)^{-1/2} = B \tag{20}$$

where B is an integration constant, an invariant, here called the **flight invariant**. The flight invariant will be shown to be the impact parameter in the following sections. Substitute $u(r) = \dot{r}/r$ back, we get

$$\frac{1}{B}r^2 n(r) = \sqrt{\dot{r}^2 + r^2} \tag{21}$$

The solution to the differential equation is

$$\int \frac{B}{r\sqrt{r^2 n\left(r\right)^2 - B^2}} dr = \theta \tag{22}$$

This 2D solution in polar coordinate is sufficient to describe all 3D ray trajectories by making a rotation of the 2D plane, which simplifies the analysis a lot.

D. Ray trajectory for the simplified refraction index

For the index of refraction in Eq (9), the integral in (20) can be rewritten as

$$\int \frac{B}{r\sqrt{r^2 + C^2 - B^2}} \mathrm{d}r = \theta \tag{23}$$

Using the appendix or an integration table, we can find the solution of Eq. (21) to be

$$r = \sqrt{B^2 - C^2} \csc \left(\frac{\sqrt{B^2 - C^2}}{B} \theta \right) \quad \text{for} \quad B > C \tag{24}$$

and

$$r = \sqrt{C^2 - B^2} \,\mathrm{csch} \left(\frac{\sqrt{C^2 - B^2}}{B} \theta \right) \quad \text{for} \quad B < C \tag{25}$$

The rays that are bent by the gravitation lens are described by Eq. (22) while the rays that are attracted by the black hole and terminate at the origin are described by Eq. (23). For both type of trajectories, when taking the limit $\theta \to 0$, the two trajectories all have

$$\lim_{\theta \to 0} y = \lim_{\theta \to 0} r \sin \theta = B, \tag{26}$$

where y is the Euclidean coordinate. This means that the flight invariant can be interpret as the impact parameter of the ray. Fig. 1 shows two categories of rays with C = 1

The following observation can be made about rays described by Eq. (22)

- A full trajectory of the ray is described for the argument of the cosecant ranging from 0 to π. Therefore, when $\frac{\sqrt{B^2 - C^2}}{B} \times 2\pi < \pi$, or equivalently $B < \frac{2}{\sqrt{3}} C$, the ray will make a closed loop around the black hole, as shown in Fig. 1

- The radius r approaches infinity as the argument of cosecant goes to π. Thus, the exiting angle θ_2 can be written as

$$\theta_2 = \frac{B\pi}{\sqrt{B^2 - C^2}} \tag{27}$$

- The exiting angle $\Delta\theta$ spanned by two rays with different flight invariant ΔB is

$$\Delta\theta = \frac{d\theta}{dB} \Delta B \approx \frac{\pi C}{\left(B^2 - C^2\right)^{3/2}} \Delta B \tag{28}$$

Here we provide a ray trajectory visualization for various values of B.

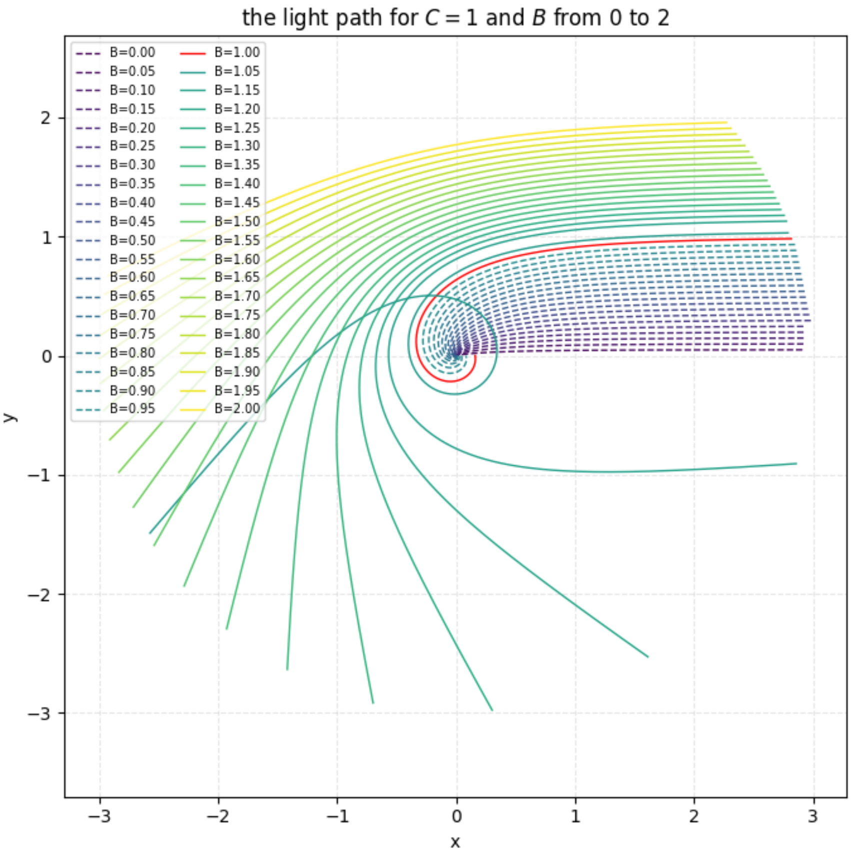


FIG. 4: Plot of various light ray trajectories corresponding to different values of B.

E. Optical Path

The optical path is given by the integral in Eq. (12), while integrating over the trajectory will be divergent, we can calculate the difference of optical path if the space-time were not curved, which converges. Simplifying the expression of F in terms of the flight invariant B

$$F = \frac{1}{B} r^2 n\left(r\right)^2 \tag{29}$$

We can rewrite the optical path difference ΔL as

$$\Delta L = \frac{1}{B} \int_0^\theta r^2 \left[n\left(r\right)^2 - 1\right] d\theta = \frac{C^2}{B}\theta \tag{30}$$

Thus, the "stretch" accumulated near a gravitational lens is finite for $B > C$. However, as the flight invariant B approaches the constant C, the optical path stretch approaches infinity, and therefore the flight time "stretches" to infinity as well.

F. Einstein Ring

In the following section, we assume the observer is on the x axis and the gravitation lens is centered at the origin.

An optical ray that starts its journey parallel to the x axis at an impact parameter $B > C$ is bent by a gravitational lens and crosses the x axis on the opposite side. The crossing of the x axis $(y = 0, \text{or } \theta = \pi)$ is given by the point

$$P_s = \left(-\sqrt{B^2 - C^2} \csc\left(\frac{\sqrt{B^2 - C^2}}{B} \pi \right), 0 \right) \tag{31}$$

By reciprocity of light trajectories (that is, by invaraince under time-reversal), the rays that originate from point P_s, which is on the x axis, will be parallel to the x axis if the gravitation lens is centered about the origin. Thus, any plane containing the x axis also contains a similar trajectory. By using a 2D trajectory and rotations around the x axis, it is possible to construct a 3D representation

$$(x, y, z) = M(\psi)\,(x, y, 0)\,, \tag{32}$$

where $M(\psi)$ is the rotation matrix around the x axis:

$$M\left(\psi\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}, \tag{33}$$

and ψ is the rotation angle about the x axis. This means that an observer on the x axis will see a ring of image of the origin star, which is the well-known Einstein ring.

G. 2D and 3D Imaging

Consider a point source P, a distant observer on the x axis. We are interested in finding where the image of the point source P will be seen by the observer if the gravitational lens is centered at the origin. A 2D case with point P whose Cartesian coordinates (x_P, y_P) is considered first. The polar coordinates of the point P are

$$(r_P, \theta_P) = \left(\sqrt{x_P^2 + y_P^2}, \arctan\left(\frac{x_P}{y_P} \right) + 2\pi k \right) \tag{34}$$

where k is an integer related to the type of bend and the number of loops traced by the optical ray around the gravitational lens. The flight invariant of the ray can be solved by substituting Eq. (34) into Eq. (24). Since there's no close form solution of the flight invariant, a numerical solution is used instead. Different solutions may be determined numerically for different values of k. The figure below illustrates a few values of k and their flight invariant.

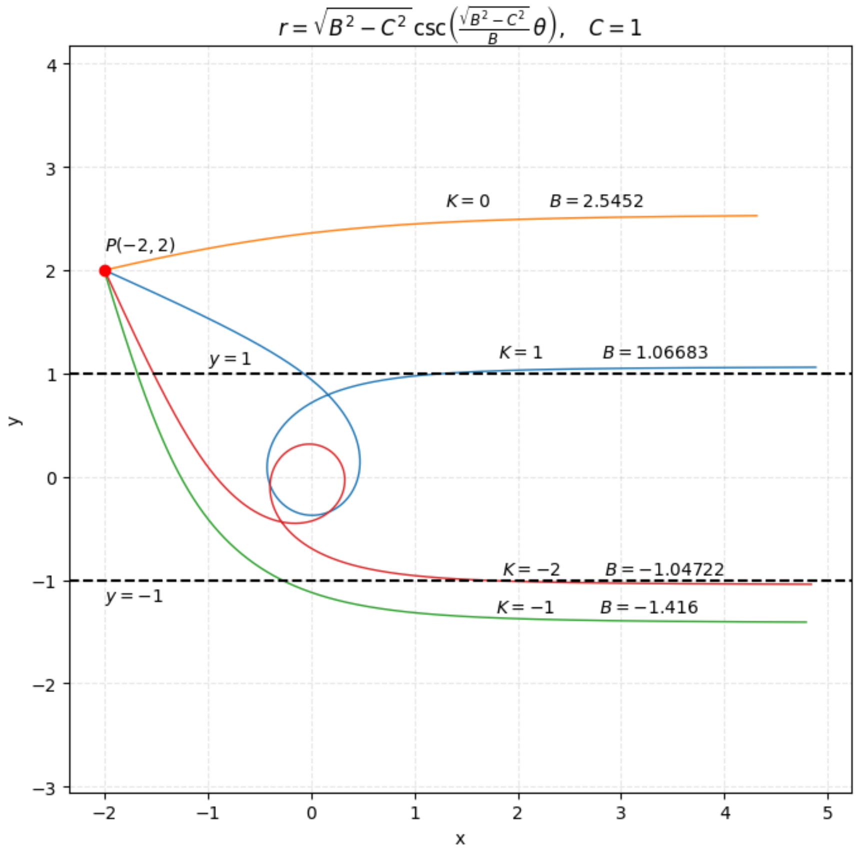


FIG. 5: Different k values and the corresponding ray trajectory around a massive body located at the origin $(0, 0)$. The light is emitted from the point P.

Now consider a point P in 3D space, the gravitational lens at the origin, and a distant observer along the positive x axis. The coordinates of the point P are $P = (x_P, y_P, z_P)$. For a spherical gradient, all trajectories are in a plane and contain the origin. Thus, the simplest way to find the 3D trajectory is to rotate the system of coordinates around the x axis to place the given point P in the xy plane and then find the 2D trajectory as described in the previous discussion. The rotation angle is

$$\psi = \arctan(y_P, z_P) \tag{35}$$

For a numerical example, consider a disk (which could be thought of as a galaxy) with radius 0.3 kpc in the sky, offset from the x axis by 0.5 kpc and located behind the gravitational lens, at $x = -3$. The Cartesian coordinates of the points on the circumference are given by

$$(x_P, y_P, z_P) = (-3, 0.5 + 0.3\cos\varphi, 0.3\sin\varphi) \tag{36}$$

where φ is a parameter ranges from 0 to 2π. The solution can be described as (B, φ) for each φ. The first few ray trajectories are shown in the figure below.

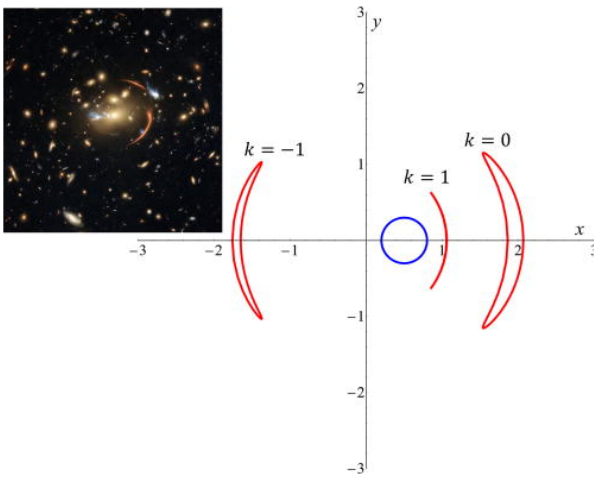


FIG. 6: First few order solutions of the 3D imaging, and an observed image of a real galaxy.

H. Ray trajectory comparison to analytic solution in General Relativity

The Schwarzschild metric yields, we can calculate various terms of the connections.

$$
\begin{aligned}
\Gamma^{\theta}_{r\theta} &= \frac{1}{r}, \\
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{r}, \\
\Gamma^{\phi}_{\theta\phi} &= \cot\theta \\
\Gamma^{t}_{rt} &= \frac{m}{r^2\left(1-\frac{2m}{r}\right)}
\end{aligned}
$$

From these Γs, we can compute the geodesic equation of the angular part

$$
\begin{aligned}
\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}u^2}\right) + 2\Gamma^{\theta}_{r\theta}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)\left(\frac{d\theta}{du}\right) + \Gamma^{\theta}_{\phi\phi}\left(\frac{d\phi}{du}\right)^2 &= 0 \\
\left(\frac{d^2\phi}{du^2}\right) + 2\Gamma^{\phi}_{r\phi}\left(\frac{dr}{du}\right)\left(\frac{d\phi}{du}\right) + 2\Gamma^{\phi}_{\theta\phi}\left(\frac{d\theta}{du}\right)\left(\frac{d\phi}{du}\right) &= 0 \\
\left(\frac{\mathrm{d}^2 t}{\mathrm{d}u^2}\right) + 2\Gamma^{t}_{rt}\left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right) &= 0
\end{aligned}
$$

where u is the parameter of the curve. Plugging in the values of Γs, the above equations can be simplified as

$$
\begin{aligned}
r\frac{\mathrm{d}^2\theta}{\mathrm{d}u^2} + 2\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}\theta}{\mathrm{d}u} - r\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}u}\right)^2 &= 0 \\
r\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} + 2\frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}\phi}{\mathrm{d}u} + 2r\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}u}\frac{\mathrm{d}\phi}{\mathrm{d}u} &= 0 \\
\frac{\mathrm{d}^2 t}{\mathrm{d}u^2} + \frac{2m}{r^2\left(1-\frac{2m}{r}\right)}\frac{\mathrm{d}t}{\mathrm{d}u}\frac{\mathrm{d}r}{\mathrm{d}u} &= 0
\end{aligned}
\tag{37}
$$

The time component is rather simple. In fact, it can be written as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\left(\left(1-\frac{2m}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}u}\right) &= 0 \\
\frac{\mathrm{d}t}{\mathrm{d}u} &= \frac{E}{1-\frac{2m}{r}}
\end{aligned}
\tag{38}
$$

If we choose the initial condition $\theta = \pi/2, \mathrm{d}\theta/\mathrm{d}u = 0$ at $u = 0$, then the θ part of Eq. (37) yields

$$
\frac{\mathrm{d}^2\theta}{\mathrm{d}u^2} = 0, \text{ at } u = 0 \tag{39}
$$

So if we choose that initial condition, θ is a constant along the trajectory. Then we can get two constants along the trajectory of the ray

$$\begin{aligned} \theta &= \frac{\pi}{2}, \\ l &= r^2 \frac{\mathrm{d}\phi}{\mathrm{d}u} \end{aligned} \tag{40}$$

where l is a relativistic analogy to the classical angular momentum, so we will refer it as the **angular momentum** of the ray. Moreover, it's convenient to consider the quantity $\mathrm{d}\phi/\mathrm{d}t$

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \frac{Er^2}{l\left(1 - \frac{2m}{r}\right)} \tag{41}$$

For the radial part, we first consider the infinitesimal invariant length.

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2 sin^2\theta \mathrm{d}\phi^2 \tag{42}$$

For light, the infinitesimal invariant length is 0, so the light trajectory is

$$0 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta \mathrm{d}\phi^2 \tag{43}$$

Divide both sides by $\mathrm{d}\phi^2$, and choose $\theta = \pi/2, \mathrm{d}\theta/\mathrm{d}\phi = 0$, we immediately get the trajectory of the ray.

$$0 = \left(1 - \frac{2m}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 - r^2 \tag{44}$$

Use the angular momentum to simplify the expressions, this finally leads to

$$\begin{aligned} \left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 &= \frac{E^2r^4}{l^2} - r^2\left(1 - \frac{2m}{r}\right) \\ &= \frac{r^4}{b^2} - r^2 + 2mr \end{aligned} \tag{45}$$

where $b = E/l$, The trajectory can be solved by numerically integrating over r

$$\phi = \int_r^\infty \frac{\mathrm{d}\rho}{\sqrt{\rho^4/b^2 - r^2 + 2mr}} \tag{46}$$

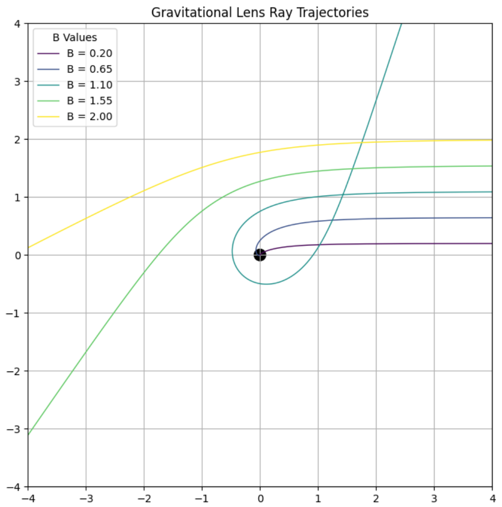


FIG. 7: Trajectory from approximate index of refraction field.

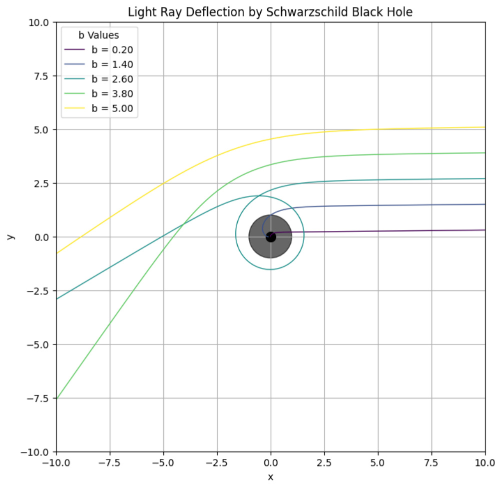


FIG. 8: Trajectory from general relativity solution.

We are interested in the region where $r \to \infty$, change the integrated variable to $u = 1/\rho$, we get

$$\begin{aligned} \phi &= \int_0^{1/r} \frac{b du}{\sqrt{1 - b^2u^2 + 2mb^2u^3}} \\ &\approx \int_0^{1/r} \frac{b du}{\sqrt{1 - b^2u^2}} \\ &= \arcsin \frac{b}{r} \end{aligned} \tag{47}$$

So our model describes the ray perfectly as $r \to \infty$ if we interpret b as B, the flight invariant. We plot the Schwarzschild solution and the approximate solution according to the paper below.

I. A Diversion into the Optical-Mechanical Analogy

1. The Theory

As mentioned above, Eq. (22) used the Fermat principle and variational calculus to derive the desired trajectory of light in a varying index of refraction field. The observant reader may have noticed a striking similarity between **Fermat's principle**,

$$\delta \int_P^Q \mathrm{d}^3 r \, n(\mathbf{r}) = 0, \tag{48}$$

and the **principle of stationary action**,

$$\delta S = \delta \int \mathrm{d}t \, L(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = 0. \tag{49}$$

We shall build on this similarity and derive a full analogy, giving us an alternative way to derive the ray trajectory equations in the paper without resorting to the Euler-Lagrange equation.

Notice that the integral in Fermat's principle is done over spatial coordinates, so we first need to transform the action integral into an integral over space. The stationary action integral is

$$\begin{aligned} S &= \int_{t_1}^{t_2} \mathrm{d}t \, L(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ &= \int_{t_1}^{t_2} \mathrm{d}t \, \left[E - V(\mathbf{r}(t), \dot{\mathbf{r}}(t), t)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \, \left[2T(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) - E\right]. \end{aligned} \tag{50}$$

Variation of a constant is identically zero, so

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} \mathrm{d}t \, 2T(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\ &= \delta \int_{t_1}^{t_2} \mathrm{d}t \, (\mathbf{v} \cdot \mathbf{p}) \\ &= \delta \int_P^Q \mathrm{d}^3 r \, |\mathbf{p}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t)| \\ &= \delta \int_P^Q \mathrm{d}^3 r \, \sqrt{2m\left(E - V(\mathbf{r}(t), \dot{\mathbf{r}}(t), t)\right)}. \end{aligned} \tag{51}$$

where we have used the fact that

$$L = E - V = 2T - E.$$

Then the quantity $\sqrt{2m\left(E - V(\mathbf{r}(t), \dot{\mathbf{r}}(t), t)\right)}$ in the stationary action variation corresponds to the refractive index $n(\mathbf{r}$ in Fermat's principle. Therefore, we have the **optical-mechanical correspondence**:

$$\boxed{E - \frac{n(\mathbf{r})^2}{2m} \Longleftrightarrow V(\mathbf{r}).} \tag{52}$$

The physical interpretation of the correspondence is that the trajectory of a light ray in a varying refractive index field obeys the same equation of motion as a particle of mass m and energy E, moving under the influence of a potential

$$V(\mathbf{r}) = E - \frac{n(\mathbf{r})^2}{2m}.$$

Therefore, we can derive the trajectory without having to resort to Fermat's principle, and thus without variational calculus.

In the historic notes section, we introduced Newton's (now obsolete) corpuscular theory of light, which treated light rays as being composed of discrete particles of nonzero mass, which travel at some finite speed. We find the theory erroneous, since it would imply light traveling at a speed of nc in a medium instead of c/n, but surprisingly the theory always gets the trajectory right! We will apply this tool to solve the problem in the paper, as an alternative way to derive the light bending equations.

2. *Solving the Light Bending Equation*

We start with the effective refractive index field

$$n(\mathbf{r})^2 = 1 + \frac{C^2}{r^2}.$$

The corresponding potential according to the optico-mechanical analogy is

$$V(\mathbf{r}) = E - \frac{1}{2m}\left(1 + \frac{C^2}{r^2}\right).$$

This is a central force problem, so angular momentum conservation gives us

$$\dot{\theta} = \frac{l}{mr^2},$$

where l is a conserved quantity we tentatively associate with some sort of angular momentum. We shall make clear its physical essence later.

Newton's equation of motion gives

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\mathrm{d}V}{\mathrm{d}r} \tag{53}$$

in the radial direction. As usual, we use the Binet transformation and substitute $u = 1/r$ into the equation. After standard calculations from Kepler's first law we get

$$\dot{r} = -\frac{l}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}, \quad \ddot{r} = -\left(\frac{lu}{m}\right)^2 \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}, \tag{54}$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{C^2 u^3}{m}. \tag{55}$$

Substituting the above into equ. (53), we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \left(1 + \frac{C^2}{l^2}\right) u = 0, \tag{56}$$

which has the solution

$$u(\theta) = \alpha e^{\sqrt{1-C^2/l^2}\theta} + \beta e^{-\sqrt{1-C^2/l^2}\theta}. \tag{57}$$

We discuss the two cases for the solution:

[1] $l > C$: the term inside the square root is positive, so the solution is hyperbolic, and we have (relabeling the constants again as α and β)

$$u(\theta) = \alpha \sinh\left(\sqrt{1 - \frac{C^2}{l^2}}\theta\right) + \beta \cosh\left(\sqrt{1 - \frac{C^2}{l^2}}\theta\right). \tag{58}$$

[2] $l < C$: the terms inside the square root is negative, so the solution is sinusoidal, and we have (relabeling the constants again as γ and δ):

$$u(\theta) = \gamma \sin\left(\sqrt{\frac{C^2}{l^2} - 1\theta}\right) + \delta \cos\left(\sqrt{\frac{C^2}{l^2} - 1\theta}\right). \tag{59}$$

It should be clear now what the quantity l holds for us: whether l exceeds C or not is the sole factor determining the "fate" of the light ray, that is, whether it shall fall into the event horizon or skim past the massive body. Therefore, we identity it with the constant B in the paper's derivation, which further puts forth the idea that the flight invariant acts like angular momentum for the light ray's motion.

Then

$$r(\theta) = \begin{cases} \sqrt{B^2 - C^2}\,\mathrm{csch}\left(\frac{\sqrt{B^2-C^2}}{B}\theta\right), & B > C, \\ \sqrt{C^2 - B^2}\,\csc\left(\frac{\sqrt{C^2-B^2}}{B}\theta\right), & B < C. \end{cases} \tag{60}$$

Therefore, we recover the previous trajectory for the light ray, without using variational calculus.

IV. SUMMARY

In summary, we use an index of refraction to mimic the effect of gravitational lens. Moreover, around the event horizon, a rather simple refraction index can be used. By Fermat's principle, we get the flight invariant, which is an important quantity defining the trajectories of a ray. Using the trajectories described by Euler-Lagrange equation, we immediately solved the image of a distant star. And finally, after some rotations, we can find out what a disk in the sky will look like described by our model, which matches the real observation.

V. APPENDIXES

A. Task Distribution

報告	簡報製作	電腦模擬	介紹、總結	歷史簡介	理論分析	資料蒐集
黃紹凱	黃紹凱	郭緯諒	郭緯諒	黃紹凱	陳景湘	全
	郭緯諒	陳景湘			林昆篁	

FIG. 9: This is the task distribution among the group members.

B. Obtaining the suitable transformation in the isotropic coordinate

The goal of this subsection is to derive the suitable transformation in the isotropic coordinate, $r = r'(1+\frac{m}{2r'})^2$. For the Schwarzschild metric we know

$$ds^2 = \left(1-\frac{m}{2r}\right)dt^2 - \left(1-\frac{m}{2r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\phi^2 \tag{61}$$

and in the isotropic coordinate we must satisfy

$$ds^2 = ar'^2dt^2 - br'^2(dr'^2 + r'^2d\theta^2 + (r')^2\sin^2\theta d\phi^2). \tag{62}$$

The associated metric tensor is given by

$$g_{\mu\nu} = \begin{pmatrix} (1-\frac{r_s}{r}) & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{63}$$

which is, interestingly, diagonal.

From above we must have these relationships:

$$\left(1-\frac{m}{2r}\right)^{-1}\mathrm{d}r^2 = b(r)'^2dr'^2,\ r^2 = br'^2r'^2 \tag{64}$$

now fist vanish the $b(r')^2$ so we get

$$\left(1-\frac{m}{2r}\right)^{-1}r'^2dr^2 = r'^2\mathrm{d}r'^2 \tag{65}$$

take the square root of both sides for positive sign we can get the ODE

$$\frac{dr}{\sqrt{r^2 - 2mr}} = \frac{dr'}{r'}$$

subject to the boundary condition $r \to \infty$ as $\rho \to \infty$, and choosing the positive root so that $\rho \to \infty$ when $r \to \infty$.

Step 1. Change of variables. Notice that

$$r^2 - 2mr = \left(r - m\right)^2 - m^2 ,$$

so that under the square-root we have a difference of squares. A standard trick is to set

$$u = r - m,$$

so that $r = u + m$ and

$$r^2 - 2mr = (u + m)^2 - 2m\,(u + m) = u^2 - m^2.$$

Hence

$$\sqrt{r^2 - 2mr} = \sqrt{u^2 - m^2} ,$$

and

$$dr = \mathrm{d}u.$$

The ODE becomes

$$\frac{du}{\sqrt{u^2 - m^2}} = \frac{dr'}{r'} .$$

Step 2. Direct integration. The left-hand side is the standard integral of the form $\displaystyle\int \mathrm{d}u/\sqrt{u^2 - m^2} = \cosh^{-1}(u/m)$, but equivalently one may recall that

$$\frac{d}{du}\left(u + \sqrt{u^2 - m^2}\right) = 1 + \frac{u}{\sqrt{u^2 - m^2}} = \frac{\sqrt{u^2 - m^2} + u}{\sqrt{u^2 - m^2}} = \frac{u + \sqrt{u^2 - m^2}}{\sqrt{u^2 - m^2}},$$

so that

$$\int \frac{du}{\sqrt{u^2 - m^2}} = \ln\!\left(u + \sqrt{u^2 - m^2}\right) + \text{constant}.$$

Thus integrating both sides of the ODE gives

$$\ln\!\left(u + \sqrt{u^2 - m^2}\right) = \ln r' + C \implies u + \sqrt{u^2 - m^2} = e^C\, r' = K\, r',$$

where $K = e^C$ is an integration constant.

Step 3. Fixing the integration constant. As $r' \to \infty$, we require $r \to \infty$, hence $u = r - m \to \infty$. In that limit $\sqrt{u^2 - m^2} \sim u$, so

$$u + \sqrt{u^2 - m^2} \;\sim\; 2u \;\sim\; 2(r - m) \;\to\; \infty.$$

On the other hand $K\,r' \to \infty$ also, so consistency in the asymptotic region demands $K = 2$. Thus

$$u + \sqrt{u^2 - m^2} \;=\; 2\,r'.$$

Step 4. Solve for r. Recall $u = r - m$, so

$$r - m \;+\; \sqrt{(r - m)^2 - m^2} \;=\; 2\,r'.$$

Rearrange to isolate the square-root,

$$\sqrt{(r - m)^2 - m^2} \;=\; 2r' - (r - m) \;=\; (r - m)\Big(\frac{2r'}{r - m} - 1\Big).$$

A more direct route is to square the relation

$$(r - m) + \sqrt{(r - m)^2 - m^2} = 2r'$$

to get

$$(r - m)^2 - m^2 \;=\; (2r' - (r - m))^2 \;=\; (r - m)^2 - 4r'(r - m) + 4r^{2\prime},$$

hence

$$-m^2 = -4r'(r - m) + 4r'^2 \quad\Longrightarrow\quad 4r'(r - m) = 4r'^2 + m^2.$$

Thus

$$r - m = r' + \frac{m^2}{4r'} \;\Longrightarrow\; r = r' + m + \frac{m^2}{4r'} = r'\Big(1 + \frac{m}{2r'}\Big)^2.$$

This is precisely

$$r \;=\; r'\Big(1 + \tfrac{m}{2r'}\Big)^2,$$

C. Simulation Code

1. *Ray trajectory from main paper*

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

C = 1.0
num_rays = 5
B_values = np.linspace(0.2,2, num_rays)
theta_range =[
    np.linspace(0.01, 3.14, 100),
    np.linspace(0.01, 3.7, 100),
    np.linspace(0.01, 7.31, 100),
    np.linspace(0.01,4 , 100),
    np.linspace(0.01, 3.15, 100)
]

def ray_trajectory(B, theta):
    if B > C:
        factor = np.sqrt(B**2 - C**2)
        r = factor / np.sin(factor * theta / B)
    else:
        factor = np.sqrt(C**2 - B**2)
        r = factor / np.sinh(factor * theta / B)
    return r

fig, ax = plt.subplots(figsize=(8, 8))
ax.set_xlim(-4, 4)
ax.set_ylim(-4, 4)
ax.set_aspect('equal')
ax.set_title("Gravitational Lens Ray Trajectories")
lens = plt.Circle((0, 0), 0.1, color='black')
```

```
ax.add_patch(lens)
ax.grid(True)

colors = plt.cm.viridis(np.linspace(0, 1, num_rays))  # colormap
lines = [ax.plot([], [], lw=1, color=colors[i], label=f'B ={B_values[i]:.2f}')[0] for i in
ax.legend(loc='upper left', title='B Values')

def init():
    for line in lines:
        line.set_data([], [])
    return lines

def update(frame):
    for i, B in enumerate(B_values):
        theta = theta_range[i][:frame + 1]
        try:
            r = ray_trajectory(B, theta)
            x = r * np.cos(theta)
            y = r * np.sin(theta)
            lines[i].set_data(x, y)
        except Exception:
            pass
    return lines

ani = FuncAnimation(fig, update, frames=len(theta_range[0]), init_func=init, blit=True, int
import HTML
HTML(ani.to_jshtml())
```

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from scipy.integrate import solve_ivp
num_rays = 5
rs = 1
b_values = np.linspace(0.2*rs,5*rs, num_rays)
u0 = 1e-5
du0 = [1 / b for b in b_values]

def geodesic(phi, y):
    u, du = y
    return [du, 1.5 * rs * u**2 - u]

#phi_min, phi_max, N = 0, 3.7, 300
phi_span = [
    (0.01, 3.14),
    (0.01, 3.31),
    (0.01, 9.9),
    (0.01, 4),
    (0.01, 3.7)
]
N=300
phi_vals = [np.linspace(*phi_span[i], N) for i in range(num_rays)]

param = {}
for i in range(num_rays):
  sol = solve_ivp(geodesic, phi_span[i], [u0, du0[i]], t_eval=phi_vals[i], rtol=1e-8, atol=
```

```
  u = sol.y[0]
  r = 1 / u
  x = r * np.cos(sol.t)
  y = r * np.sin(sol.t)
  param[i] = (x, y)

fig, ax = plt.subplots(figsize=(8,8))
ax.set_aspect('equal')
ax.set_xlim(-10, 10)
ax.set_ylim(-10, 10)
ax.grid(True)
ax.set_title("Light Ray Deflection by Schwarzschild Black Hole")
ax.set_xlabel("x")
ax.set_ylabel("y")

plt.plot(0, 0, 'ko', markersize=10)
horizon = plt.Circle((0, 0), rs, color='black', alpha=0.6)
plt.gca().add_patch(horizon)

colors = plt.cm.viridis(np.linspace(0, 1, num_rays))  # colormap
lines = [ax.plot([], [], lw=1, color=colors[i], label=f'b = {b_values[i]:.2f}')[0] for i in
ax.legend(loc='upper left', title='b Values')

def init():
    for line in lines:
        line.set_data([], [])
    return lines

def update(i):
    for j in range(num_rays):
```

```
        x, y = param[j]
        lines[j].set_data(x[:i], y[:i])
    #dot.set_data(x[i], y[i])
    return lines# dot

ani = FuncAnimation(fig, update, frames=300, init_func=init, blit=True, interval=10)

from IPython.display import HTML
HTML(ani.to_jshtml())
```

3. Rays emerging from one source

```
import numpy as np
import matplotlib.pyplot as plt

def plot_csc_xy(B_values, Kvalues,C=1.0):
    plt.figure(figsize=(7, 7))
    for i in range(len(B_values)):
        K=Kvalues[i]
        B=B_values[i]
        if (B > 0):
            amplitude = np.sqrt(max(0, B**2 - C**2))
            factor = amplitude / B if B != 0 else 0
            if  (i<4):
                theta = np.linspace(0.0001, np.pi / factor, 3000)
            elif(i<=7):
                theta = np.linspace(2*np.pi / factor,3*np.pi / factor, 3000)
            else:
                theta = np.linspace((-2)*np.pi / factor,(-1)*np.pi / factor, 3000)
```

```
        elif (B < 0) :
            amplitude = np.sqrt(max(0, B**2 - C**2))
            factor = amplitude / B if B != 0 else 0
            if  (i<4):
                theta = np.linspace(np.pi / factor, 0.0001, 3000)
            elif(i<=7):
                theta = np.linspace(3*np.pi / factor,2*np.pi / factor, 3000)
            else:
                theta = np.linspace((-1)*np.pi / factor,(-2)*np.pi / factor, 3000)
        else:
            continue
        with np.errstate(divide='ignore', invalid='ignore'):
            r = amplitude / np.sin(factor * theta)
        r[np.abs(r) > 5] = np.nan  # mask asymptotes

        x = r * np.cos(theta)
        y = r * np.sin(theta)

        mask = x >= -2
        x = x[mask]
        y = y[mask]

        if (B > 0):
            if (i==4):
                plt.text(x[-1]+0.1, y[-1], f'B={B}', fontsize='small')
            elif (i<11):
                plt.text(x[1]-2, y[1]+0.05 , f'$B={B}$', fontsize=10)
                plt.text(x[1]-3, y[1]+0., f'$K={K}$', fontsize=10)
            else:
```

```
            plt.text(x[-1]+0.1, y[-1] , f'B={B}', fontsize='small')
    if (B < 0):
        if  (i<4):
            plt.text(x[-1]-2, y[-1]+0.1 , f'$B={B}$', fontsize=10)
            plt.text(x[-1]-3, y[-1]+0.1, f'$K={K}$', fontsize=10)
        elif (6<i<9) :
            plt.text(x[-1]-0.1, y[-1]-0.2 , f'B={B}', fontsize='small')
        else:
            plt.text(x[1]+0.2, y[1]+0.3 , f'B={B}', fontsize='small')
    if (B > 0):
        plt.plot(x, y, linewidth=1.0, label=f'B={B:.5f}')

    elif (B < 0):
        plt.plot(x, y, linewidth=1.0, label=f'B={B:.5f}')

plt.plot(-2, 2, 'ro', label='(-2, 2)')
plt.title(r"$r = \sqrt{B^2 - C^2}\,\csc\!\left("r"\frac{\sqrt{B^2 - C^2}}{B}\,\theta"r"

plt.axhline(y=1, color='black', linestyle='--', label='y = 1')
plt.axhline(y=-1, color='black', linestyle='--', label='y = -1')
plt.text(-1, 1.1, f'$y=1$', fontsize=10)
plt.text(-2, -1.2, f'$y=-1$', fontsize=10)
plt.text(-2, 2.2, f'$P(-2,2)$', fontsize=10)
plt.xlabel("x")
plt.ylabel("y")
plt.axis("equal")
plt.grid(True, linestyle="--", alpha=0.3)
plt.tight_layout()
plt.show()
```

```
def main():
    B_values = [ 1.06683, 2.5452, -1.416, -1.04722]#[1.598383,2.910835,-1.30869,-1.96239,-1
    K_values = [1,0,-1,-2]

    plot_csc_xy(B_values, K_values,C=1.0)

if __name__ == "__main__":
main()
```

4. Ray trajectories for varying B

```
import numpy as np
import matplotlib.pyplot as plt

def plot_csc_xy(B_values, C=1.0):
    plt.figure(figsize=(7, 7))
    cmap = plt.cm.viridis
    colors = cmap(np.linspace(0, 1, len(B_values)))

    for i, B in enumerate(B_values):
        if B > 1:
            amplitude = np.sqrt(max(0, B**2 - C**2))
            factor = amplitude / B if B != 0 else 0

            theta = np.linspace(0.0001, np.pi / factor, 3000)
            with np.errstate(divide='ignore', invalid='ignore'):
                r = amplitude / np.sin(factor * theta)
            r[np.abs(r) > 3] = np.nan

```

```
        x = r * np.cos(theta)
        y = r * np.sin(theta)
        plt.plot(x, y, color=colors[i], linewidth=1.0, label=f'B={B:.2f}')

    elif B < 1:
        amplitude = np.sqrt(max(0, C**2 - B**2))
        factor = amplitude / B if B != 0 else 0

        theta = np.linspace(0.0001, 2 * np.pi, 3000)
        with np.errstate(divide='ignore', invalid='ignore'):
            r = amplitude / np.sinh(factor * theta)
        r[np.abs(r) > 3] = np.nan

        x = r * np.cos(theta)
        y = r * np.sin(theta)
        plt.plot(x, y, color=colors[i], linewidth=1.0, linestyle='--', label=f'B={B:.2f
    if B == 1:

        theta = np.linspace(0.0001, 2*np.pi, 3000)
        with np.errstate(divide='ignore', invalid='ignore'):
            r = 1/theta
        r[np.abs(r) > 3] = np.nan

        x = r * np.cos(theta)
        y = r * np.sin(theta)
        plt.plot(x, y,  linewidth=1.0, label=f'B={B:.2f}',c='r')

plt.title("the light path for $C=1$ and $B$ from $0$ to $2$")
plt.xlabel("x")
plt.ylabel("y")
```

```python
    plt.axis("equal")
    plt.grid(True, linestyle="--", alpha=0.3)
    plt.legend(loc="upper left", ncol=2, fontsize=7)
    plt.tight_layout()
    plt.show()

def main():
    B_values = np.arange(0, 2.00 + 1e-8, 0.05)
    B_values = B_values[B_values != 1.1]
    plot_csc_xy(B_values, C=1.0)
```

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