# 相對論期末報告
# A simple model of a gravitational lens from geometric optics

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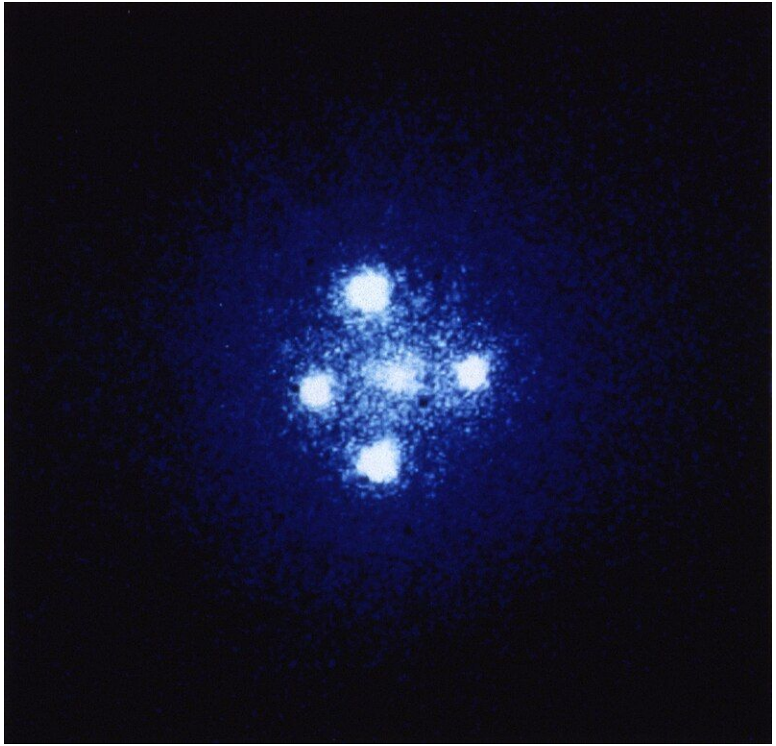
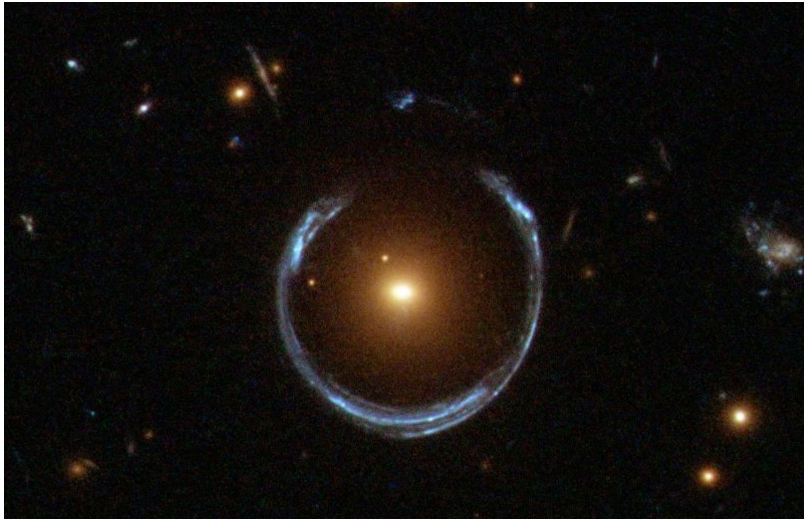
日期：2025 Mar. 25

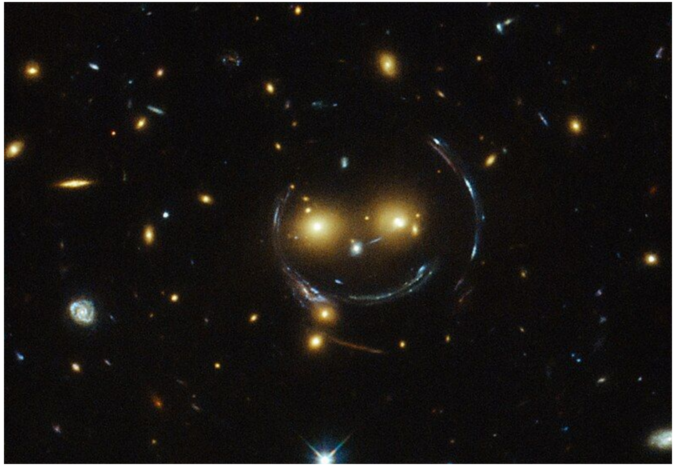
授課教師：蔣正偉

# 什麼是重力透鏡？

# What is Gravitational Lensing?







ESA/Hubble & NASA derivative work: Bulwersator
(left) Hubble sees a smiling lens / (right) NASA

# 重力透鏡簡史

1700s：光在重力場中發生偏折（Sir I. Newton 光粒子說, H. Cavendish）

**(Corpuscular Theory of Light)** Light is made up of discrete particles (corpuscles) which travel in a straight line with a finite velocity and momentum.

1804：計算光在重力場中的偏折（J. G. von Soldner）

1912：愛因斯坦環（A. Einstein）

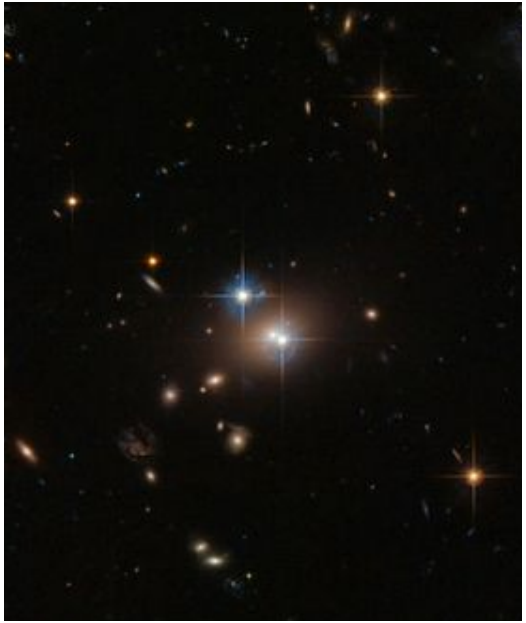
1919：測量光的偏折（Sir A. Eddington & Sir F. W. Dyson）

# 重力透鏡簡史

1937：星系團可以實現重力透鏡（Fritz Zwicky）

1979：觀測到重力透鏡效應（Dennis Walsh et al.）

1979～重力透鏡成為研究暗物質的重要媒介



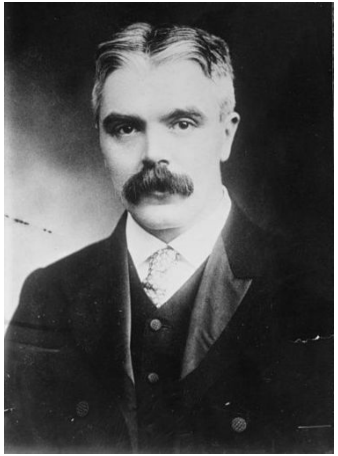
*The Twin Quasar, QSO 0957+561 A/B*

ESA/Hubble Picture of the Week.

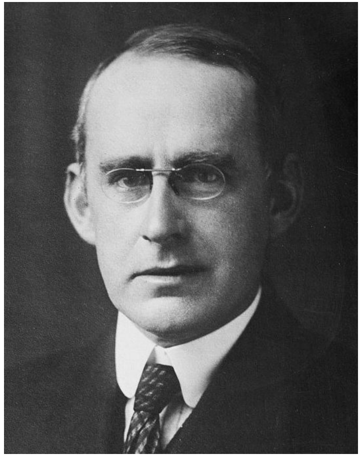
# How do (did) we know GR is right?

- 愛丁頓實驗：廣義相對論的三大經典檢驗之一
- 光發生偏折

$$\alpha = \sqrt{\frac{GM}{c^2 b}} \longrightarrow \alpha = \sqrt{\frac{4GM}{c^2 b}}$$



*Sir Frank W. Dyson*



*Sir Arthur Eddington*

**愛因斯坦及廣義相對論因此聲名遠播！**

(left) Bain News Service. (right) George Grantham Bain Collection.

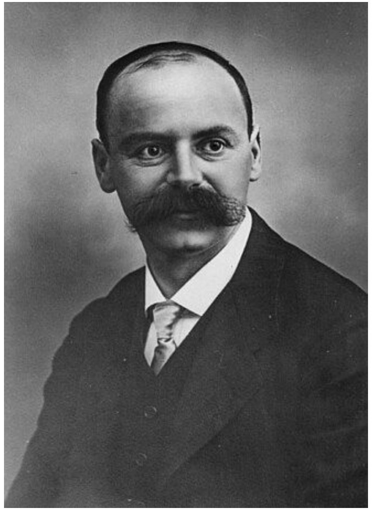
# 廣義相對論模型 (1/2)

- 史瓦西度規（K. Schwarzschild, 1915）:

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

球對稱、不轉動、不帶電荷

- 史瓦西半徑以內是**事件視界**
- 1.5 史瓦西半徑處是**光子球層**



*Karl Schwarzschild*

# 廣義相對論模型 (2/2)

- 重力場可以視為空間有等效折射率 *n*

$$n(r)^2 = \left(1 + \frac{m}{2r}\right)^6 \left(1 - \frac{m}{2r}\right)^{-2}, \quad m = \frac{GM}{c^2}$$

- 漸近 : *r* 接近 *m/2* 時有一個奇異點

Am. J. Phys. 92, 878–884 (2024). *A simple model of a gravitational lens from geometric optics*.

# 幾何光學模型 (2/4)

- 相同漸近行為的簡易模型

$$n(r)^2 = 1 + \frac{C^2}{r^2}, \quad C = 4m.$$

$r \rightarrow r - m/2$ 接近 0 時有奇異點

- 變分法

$$L = \int \mathrm{d}\theta \, F(\theta, r(\theta)\, \dot{r}(\theta)), \quad F = n(r)\sqrt{\dot{r}^2 + r^2}$$

**費馬原理**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

# 幾何光學模型 (3/4)

- 飛行不變量(flight invariant)決定光的命運
- 對應於古典角動量 $$rn(r)\left(1+u(r)^2\right)^{-1/2}=B \iff L=mr^2\dot{\theta}$$
- 預期結果:

  角動量太大 → 掠過黑洞外圍

  角動量太小 → 墜入黑洞事件視界

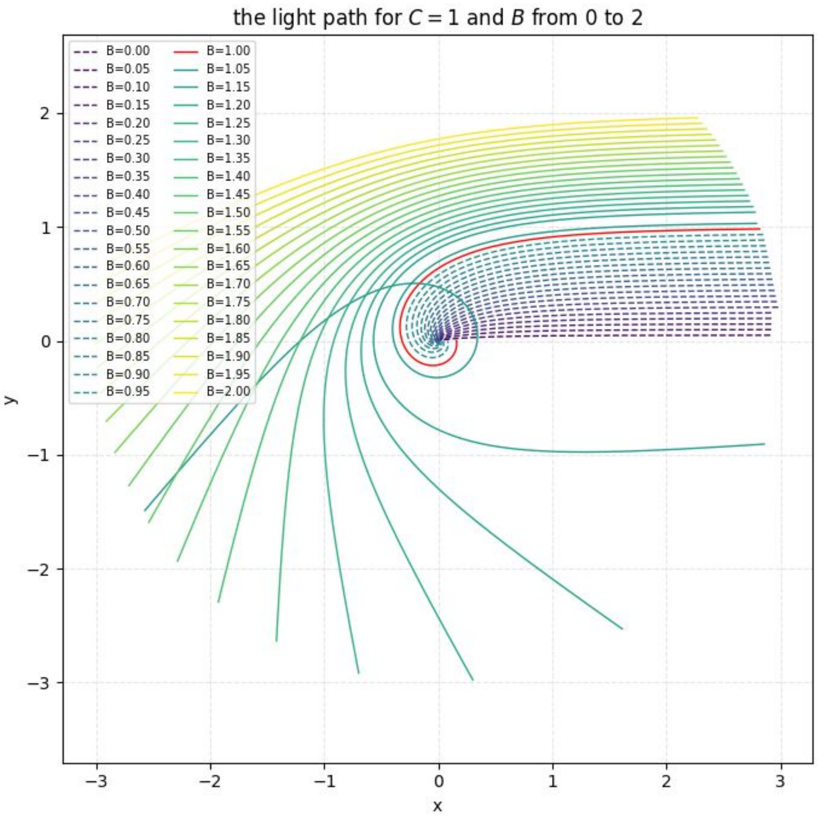
# 幾何光學模型 (4/4)

- 路徑解:

$$r = \sqrt{C^2 - B^2}\ \operatorname{csch}\left(\frac{\sqrt{C^2 - B^2}}{B}\theta\right) \quad \text{for} \quad B < C$$

$$r = \sqrt{B^2 - C^2}\csc\left(\frac{\sqrt{B^2 - C^2}}{B}\theta\right) \quad \text{for} \quad B > C$$

$$rn(r)\left(1 + u(r)^2\right)^{-1/2} = B$$



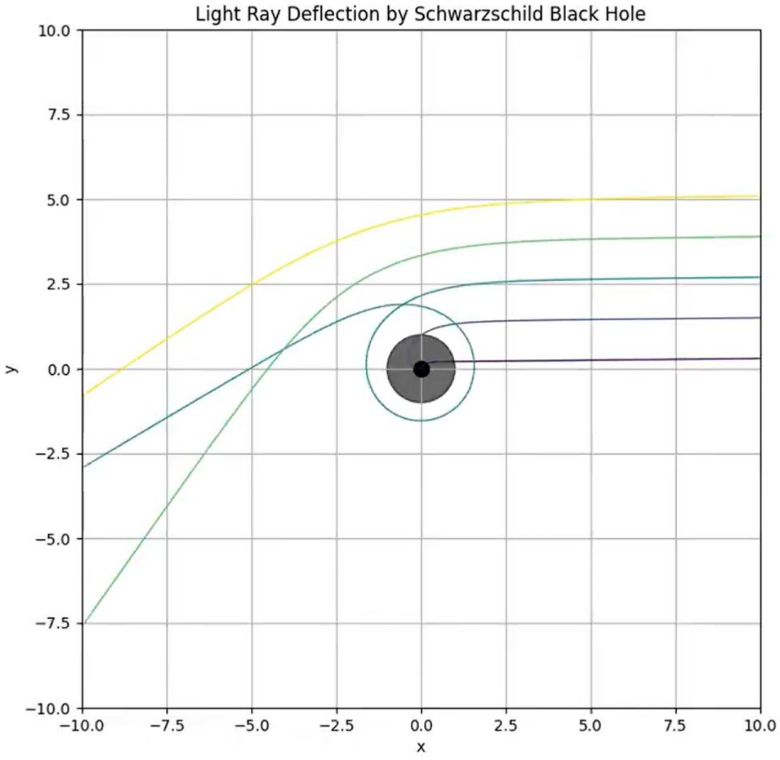
# 模擬展示：Toy model vs. analytic solution

- 掠過黑洞的史瓦西解析解由 ODE 給出

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4}{b^2} - r^2 + 2Mr$$

- 軌跡可以寫出積分形式 → 數值積分

$$\phi(r) = \int_{r_0}^{r} \frac{\mathrm{d}\rho}{\sqrt{(\rho^4/b^2) - \rho^2 + 2M\rho}}$$



# 光學-機械類比：光粒子說的復活 (1/3)

- 光學-機械類比（Optico-mechanical analogy）可以推導出相同的光軌跡

（費馬原理） （最小作用量原理）

$$\delta \int_P^Q \mathrm{d}^3 r\, n(\mathbf{r}) = 0 \quad \Longleftrightarrow \quad \delta \int_P^Q \mathrm{d}^3 r \sqrt{2m\left(E - V(\mathbf{r})\right)} = 0$$

$$n(\mathbf{r}) \quad \Longleftrightarrow \quad \sqrt{2m\left(E - V(\mathbf{r})\right)}$$

# 光學-機械類比：光粒子說的復活 (2/3)

$$\delta S = \delta \int \mathrm{d}t\, L(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) = 0.$$

（最小作用量原理）

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} \mathrm{d}t\, 2T(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \\
&= \delta \int_{t_1}^{t_2} \mathrm{d}t\, (\mathbf{v} \cdot \mathbf{p}) \\
&= \delta \int_P^Q \mathrm{d}^3 r\, |\mathbf{p}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t)| \\
&= \delta \int_P^Q \mathrm{d}^3 r\, \sqrt{2m\,(E - V(\mathbf{r}(t), \dot{\mathbf{r}}(t), t))}.
\end{aligned}$$

# 光學-機械類比：光粒子說的復活 (3/3)

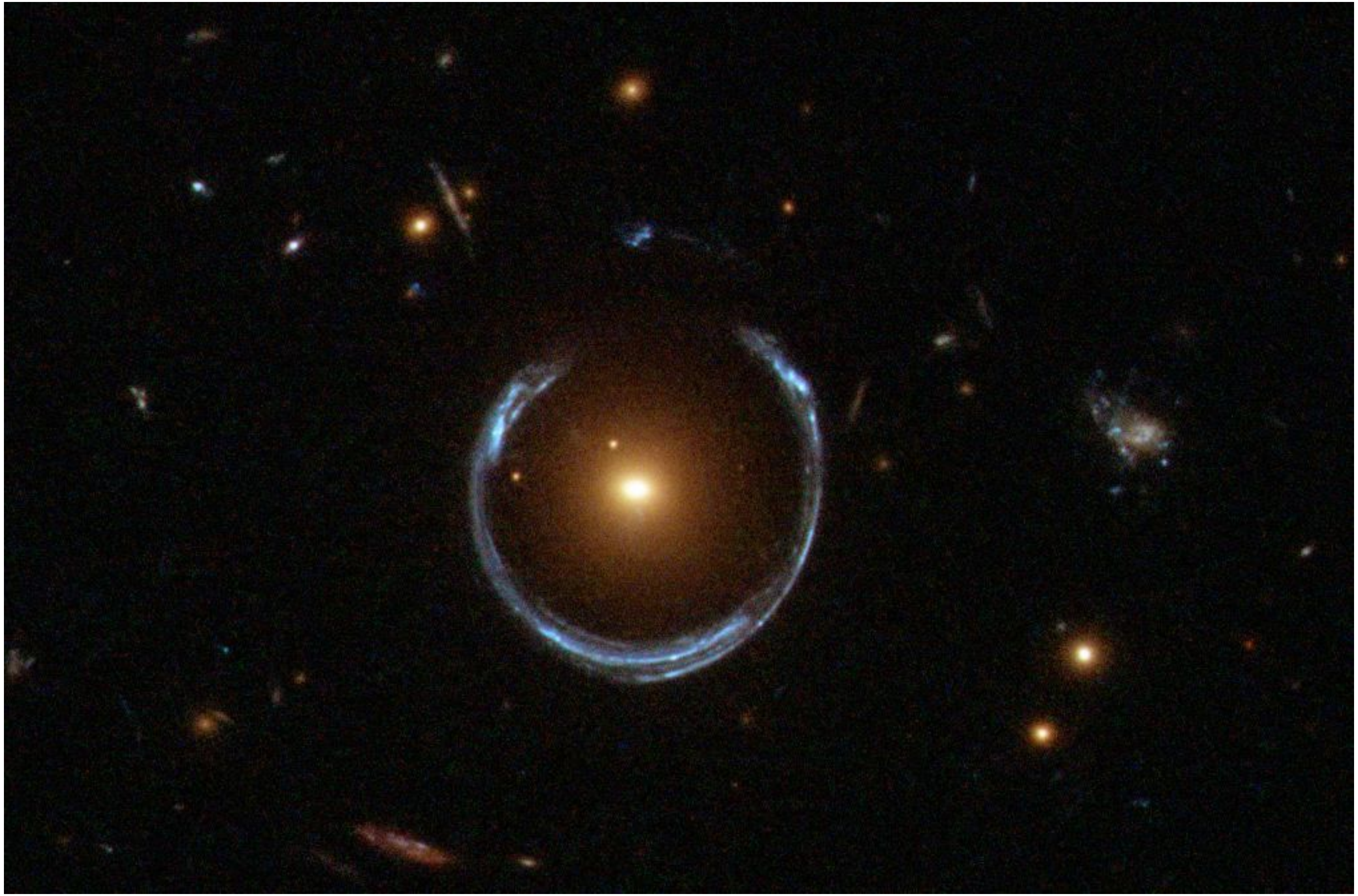
Claim: 變折射率 *n* 介質中的光線，其軌跡與位場 *V* 中能量 *E*、質量 *m* 的質點運動軌跡相同

$$V(\mathbf{r}) = E - \frac{n(\mathbf{r})^2}{2m}$$

- 與論文相同的解

$$r = \sqrt{C^2 - B^2}\ \operatorname{csch}\left(\frac{\sqrt{C^2 - B^2}}{B}\theta\right) \quad \text{for} \quad B < C$$

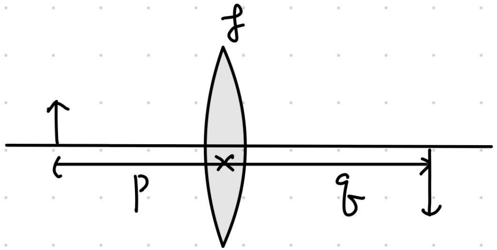
$$r = \sqrt{B^2 - C^2}\csc\left(\frac{\sqrt{B^2 - C^2}}{B}\theta\right) \quad \text{for} \quad B > C$$



ESA/Hubble & NASA derivative work: Bulwersator

# 愛因斯坦環

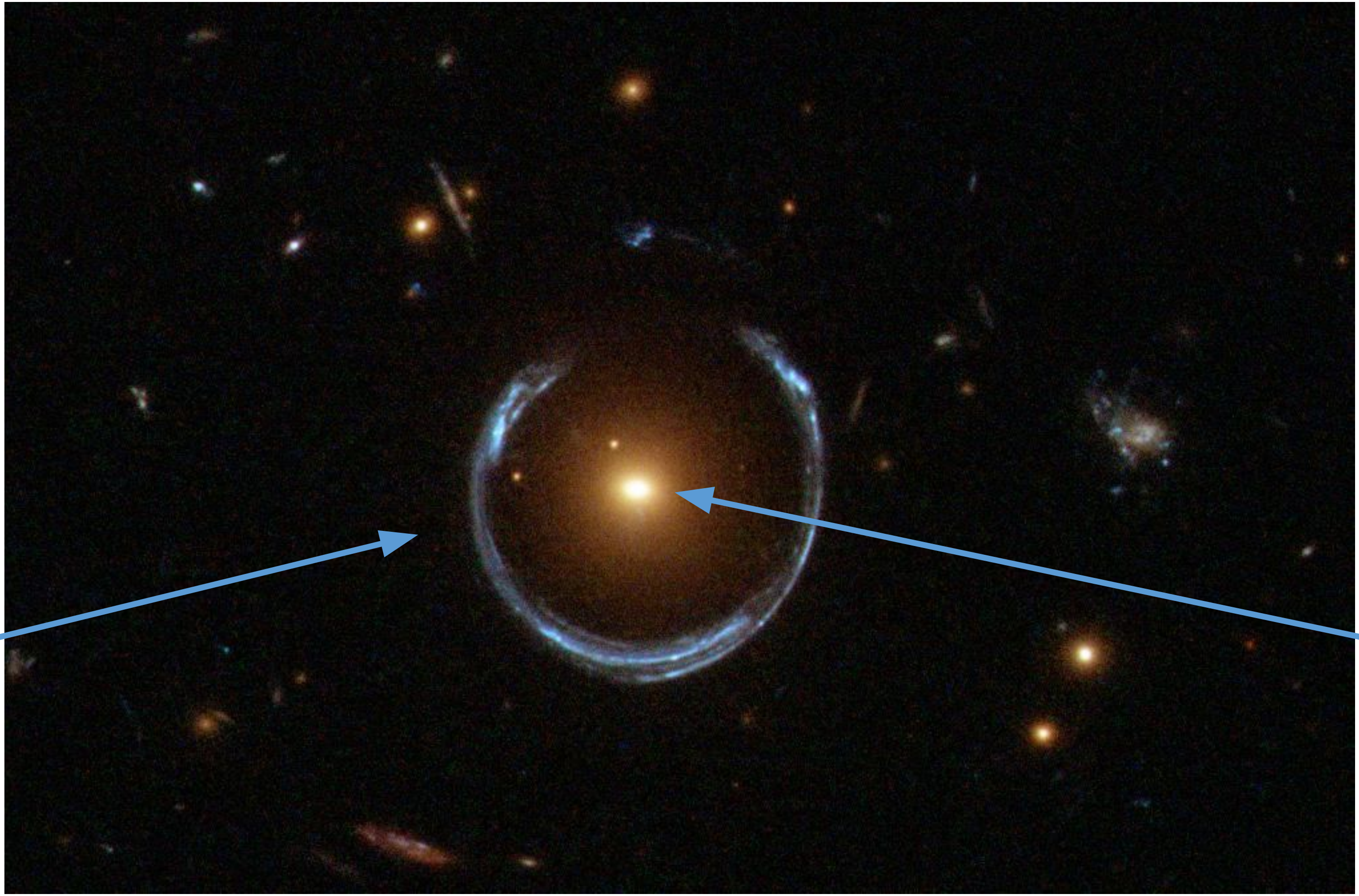
- 成因：近處的恆星和暗物質
- 愛因斯坦環角半徑： $\theta_{\mathrm{E}} = \sqrt{\frac{4GM}{c^2}\frac{D-\delta}{D\delta}}$ (A. Einstein, 1912)




$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

- 此系統即為焦距 = 撞擊參數 $b$ 的光學透鏡系統

$$\frac{D-\delta}{D\delta} = \frac{1}{\delta} - \frac{1}{D} = \frac{1}{f}$$



遠方恆星

近處恆星

ESA/Hubble & NASA derivative work: Bulwersator

# 二維成像

- 遠方共線星體形成愛因斯坦環
- 角半徑跟距離正相關

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D-\delta}{D\delta}}$$

- 繞行次數造成不同成像

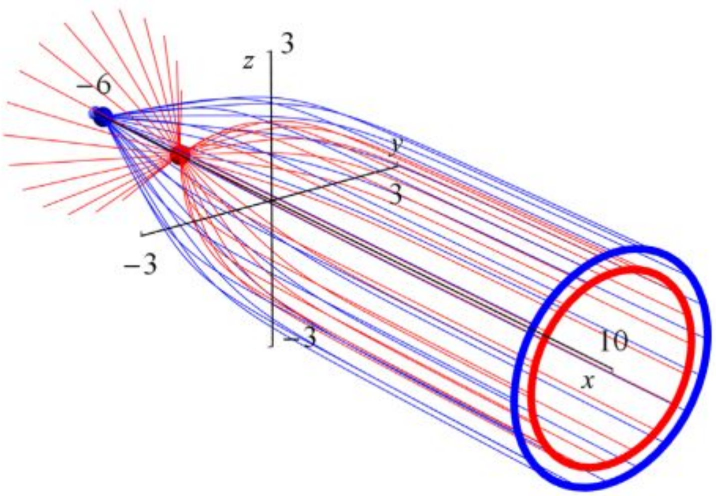


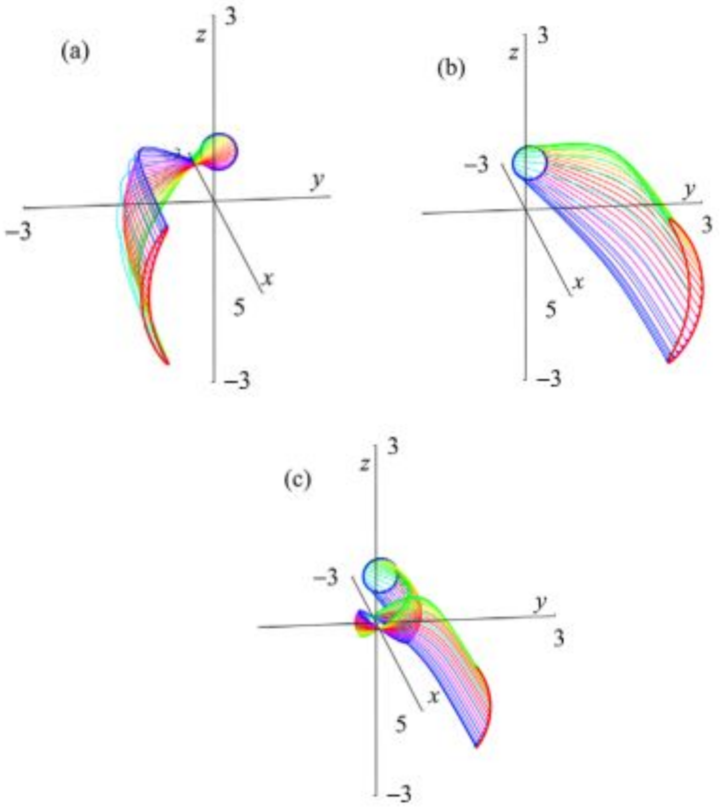

Fig. 3. Two ray bundles showing a formation of the Einstein rings by a gravitational lens at the origin. Two point sources (stars) are on the $x$ axis on the side opposite the observer. The constant $C=1$. The bend angles are $\pi/8$ and $\pi/5$.

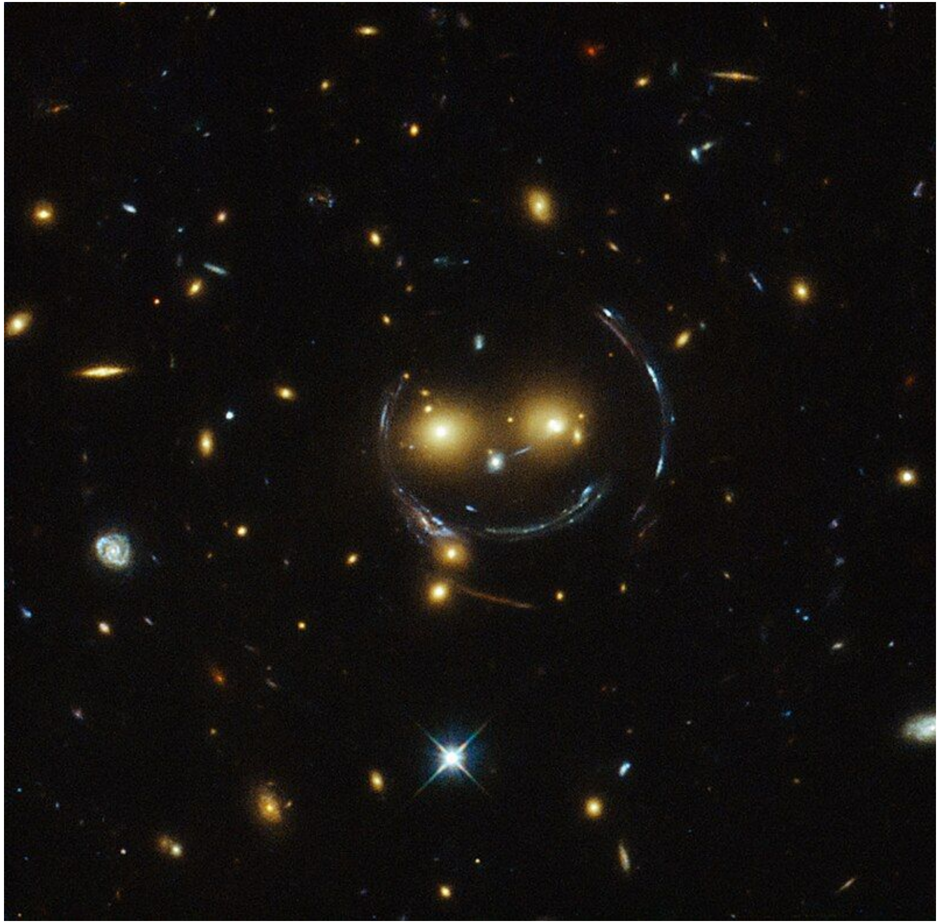
# 三維成像

- 圖形出現扭曲
- 圖片對應到不同階成像

$$(r_P, \theta_P) = \left(\sqrt{x_P^2 + y_P^2}, \arctan(x_P, y_P) + 2\pi k\right)$$

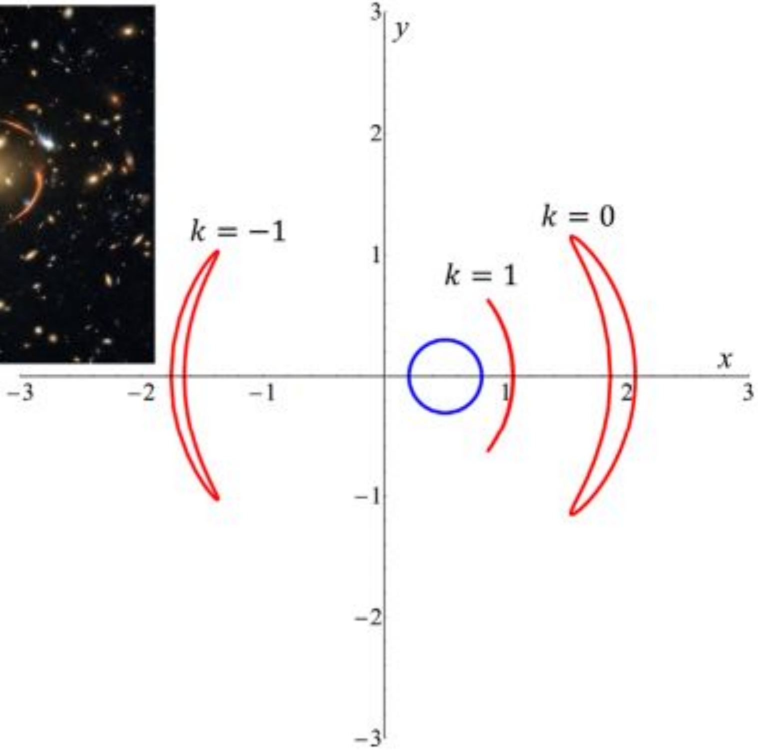
- **弧狀星系**影像是真實的物理現象





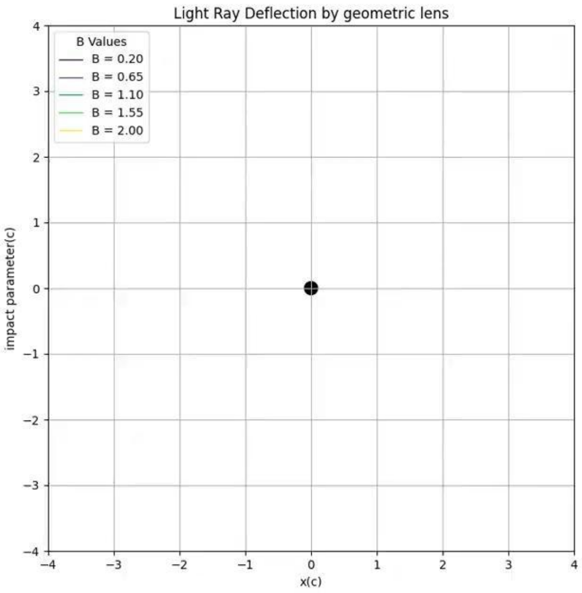
http://www.nasa.gov/content/hubble-sees-a-smiling-lens/

# 弧状星系

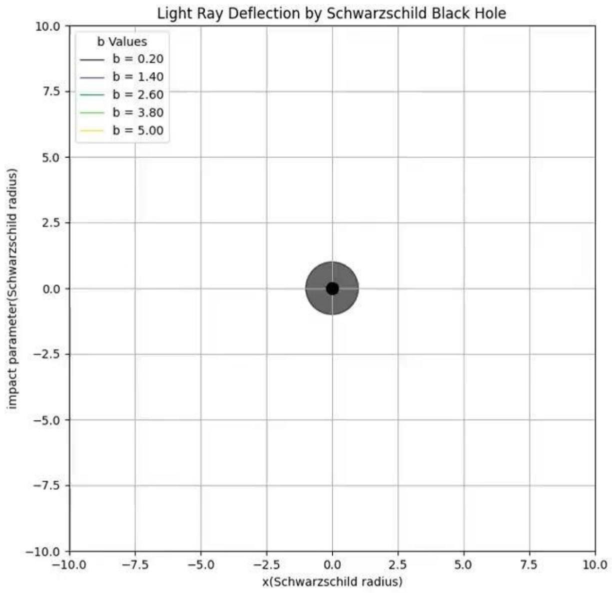


# 光路圖模擬展示

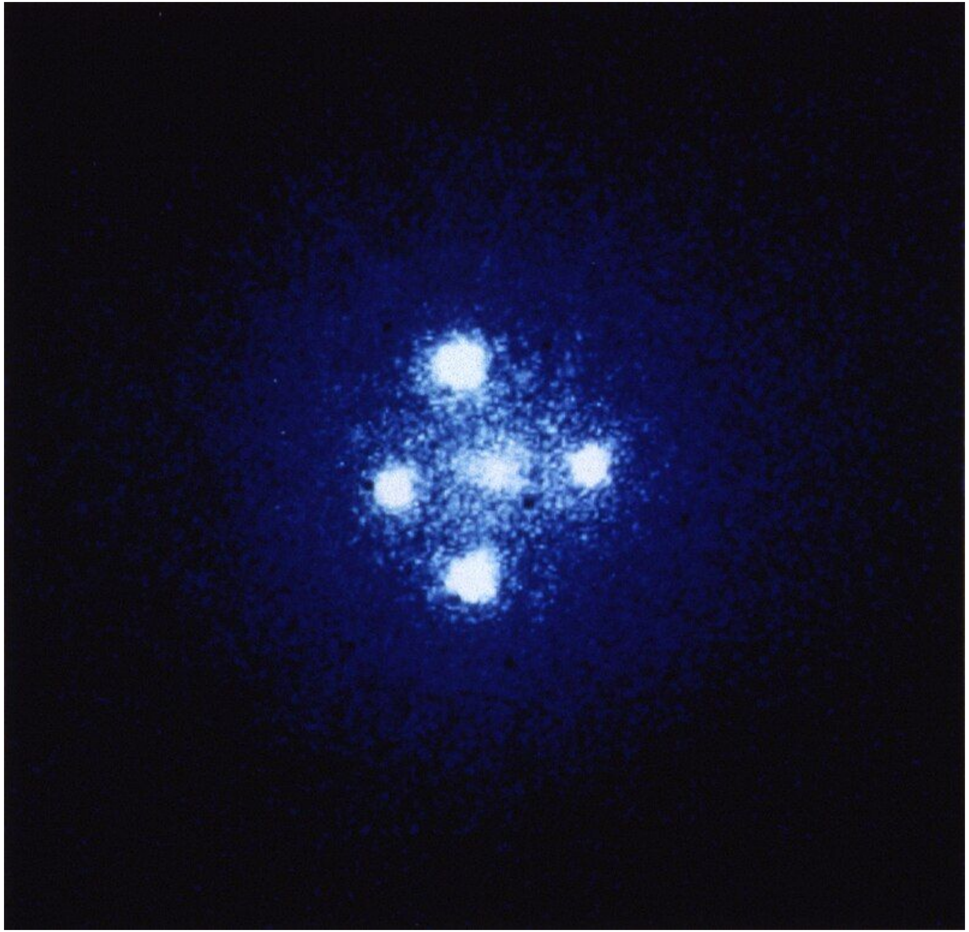
## Geometric optics



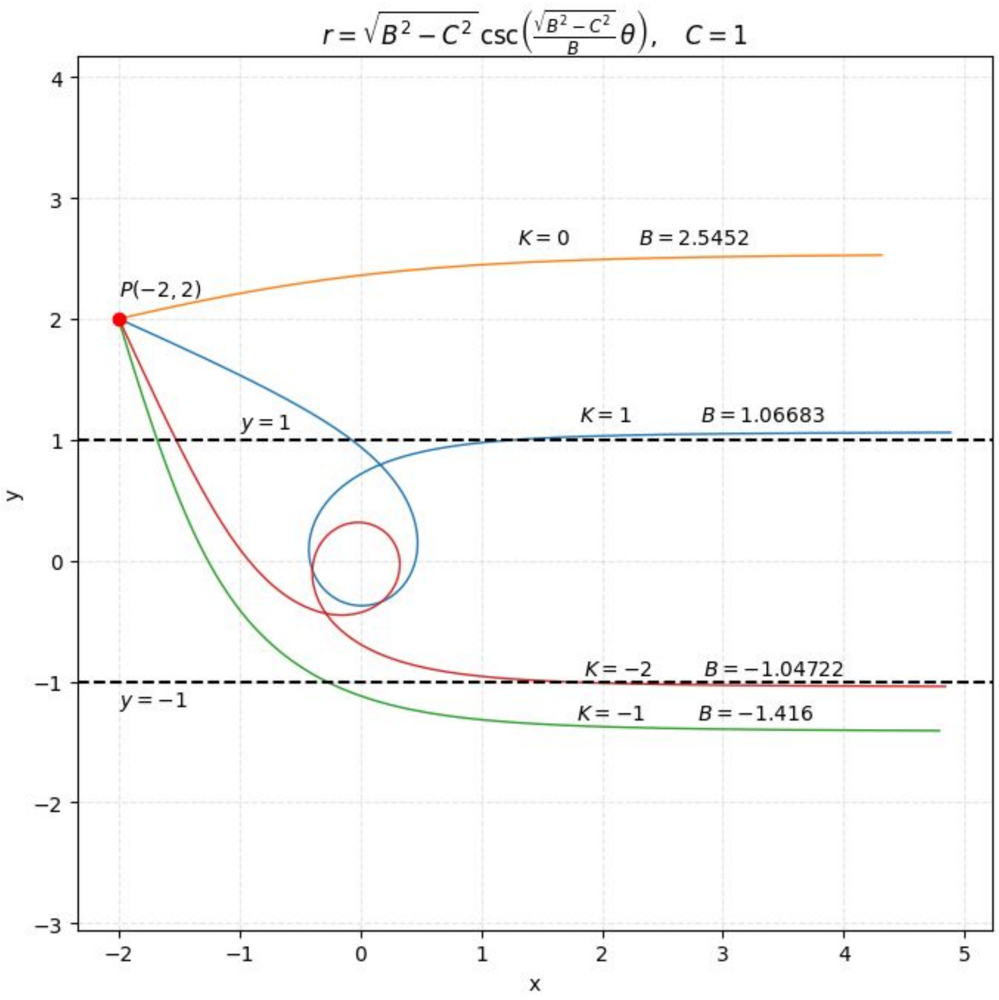

## Schwarzschild lens




This is our code!



NASA, ESA, and STScI



$r = \sqrt{B^2 - C^2}\,\csc\left(\frac{\sqrt{B^2-C^2}}{B}\theta\right), \quad C = 1$
$K = 0$ $B = 2.5452$
$P(-2, 2)$
$K = 1$ $B = 1.06683$
$y = 1$
$K = -2$ $B = -1.04722$
$y = -1$
$K = -1$ $B = -1.416$
x
y

# 結論

- 這個理論跟廣義相對論有什麼關係？

沒有關係... 但是定性解釋許多廣義相對論效應

- 透過費馬原理或光-力類比推導光線軌跡
- **定性模擬**史瓦西**重力透鏡**的效應
- 預測**光線彎曲**、**愛因斯坦環**以及**重影**等天文觀測影像

# 參考資料

[1] (Main paper) Bogdan Szafraniec; James F. Harford. ***A simple model of a gravitational lens from geometric optics***. *Am. J. Phys. 92, 878–884 (2024).*

[2] (Historical Notes) ***The 1919 eclipse results that verified general relativity and their later detractors: a story re-told***

[3] (Corpuscular Theory of Light) ***Opticks: or, A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light***

# 參考資料


*ESA/Hubble & NASA derivative work: Bulwersator*

*NASA: Hubble sees a smiling lens*

(Einstein Cross) *NASA, ESA, and STScI*

*Bill Keel's WWW Gallery-Active Galaxies and Quasars*

*ESA/Hubble Picture of the Week.*

*(Picture of Sir Dyson) Bain News Service.*

*(Picture of Sir Eddington) George Grantham Bain Collection.*

*Wikipedia - Twin Quasar* *https://en.wikipedia.org/wiki/Twin_Quasar*

(simulation code)
https://colab.research.google.com/drive/1tadNsR09vGKYFzjEemNTBonMyKSAx2kW?usp=sharing

# 分工表

| | | | | | | |
|---|---|---|---|---|---|---|
| 報告 | 簡報製作 | 電腦模擬 | 介紹、總結 | 歷史簡介 | 理論分析 | 資料蒐集 |
| 黃紹凱 | 黃紹凱 | 郭緯諒 | 郭緯諒 | 黃紹凱 | 陳景湘 | 全 |
| | 郭緯諒 | 陳景湘 | | | 林昆篁 | |

Thank You !

Any Questions?

# Equations

$$\theta_\text{E} = \sqrt{\frac{4GM}{c^2}\frac{D\delta}{D-\delta}}$$

$$\alpha = \sqrt{\frac{4GM}{c^2b}}$$

$$\alpha = \sqrt{\frac{GM}{c^2b}}$$

$$\delta \int^Q_P \mathrm{d}^3r\, n(\mathbf{r}) = 0 \quad \Longleftrightarrow \quad$$

$$\delta\int^Q_P\mathrm{d}^3r\,\sqrt{2m\left(E-V(\mathbf{r})\right)}= 0$$

$$n(\mathbf{r}) = 0$$

# Equations

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4}{b^2} - r^2 + 2Mr$$

$$\phi(r) = \int^r_{r_0} \frac{\mathrm{d}\rho}{\sqrt{(\rho^4 / b^2) - \rho^2 + 2M\rho}}$$