歐拉方程式 The Euler Equations

$$\begin{aligned}\tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2,\\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3,\\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1.\end{aligned}$$

$$\frac{d\mathbf{L}}{dt} = \frac{\delta \mathbf{L}}{\delta t} + \boldsymbol{\omega} \times \mathbf{L}.$$

旋轉坐標系中向量隨時間的變化公式：

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{r}} + \boldsymbol{\Omega}\times\right]\boldsymbol{f}$$

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3).$$

I 代表物理時靜止的性質，力矩是物體移動時的性質
[HW] 把上式展開得到歐拉方程組

Sol. 把外積展開，各個分量對應在一起

https://en.wikipedia.org/wiki/Tennis_racket_theorem

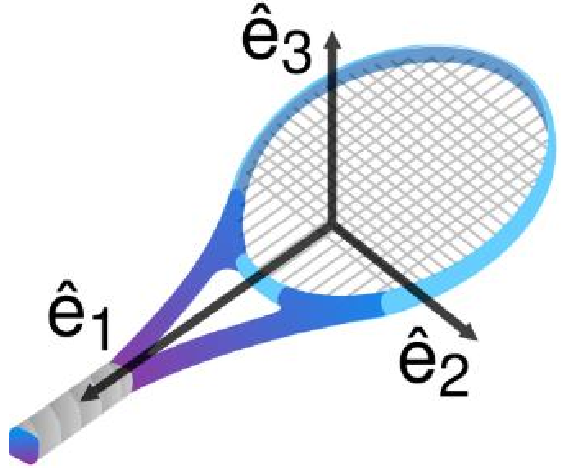
中間軸定理（網球拍定理）

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2,$$

$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3,$$

$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1.$$

相對於轉動坐標系的角速度



Thm. (Intermediate axis) 物體繞中間軸轉動時運動不穩定。

Sol. **[HW]** 完成繞 2 跟 3 軸轉的分析

假設 $I_1 > I_2 > I_3$，此時中間軸是 I_2 對應的主軸方向（$\boldsymbol{\omega}_2$）。繞 1 轉時，$\omega_2(0)$、$\omega_3(0)$ 很小。根據 1 式，$\dot{\omega}_1 \approx 0$，所以 ω_1 大約是定值。

$$0 = I_2\ddot{\omega}_2 + (I_1 - I_3)\omega_1\dot{\omega}_3 = I_2\ddot{\omega}_2 + (I_1 - I_3)\omega_1\left(\frac{I_1 - I_2}{I_3}\omega_2\omega_1\right)$$

$$\Rightarrow \ddot{\omega}_2 + \left[\frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2\right]\omega_2 = 0$$

$$\Rightarrow \ddot{\omega}_3 + \left[\frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2\right]\omega_3 = 0$$

擾動的進動頻率：

$$\Omega^2 = \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2$$

https://en.wikipedia.org/wiki/Tennis_racket_theorem

歐拉角 Euler angles

歐拉角的轉法：
繞 z 轉、繞 x' 轉、繞 z' 轉

$$\mathbf{A} = \mathbf{B}\mathbf{C}\mathbf{D}$$

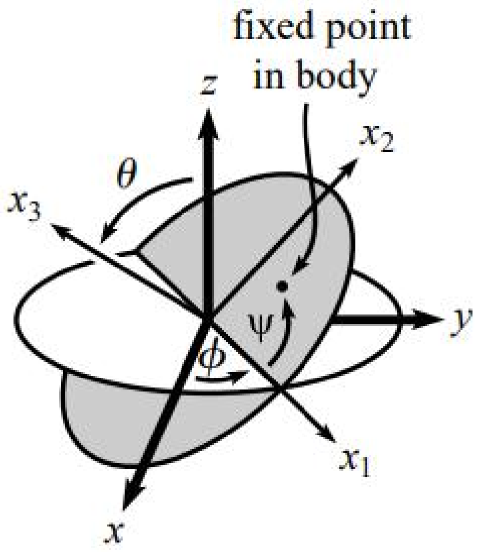
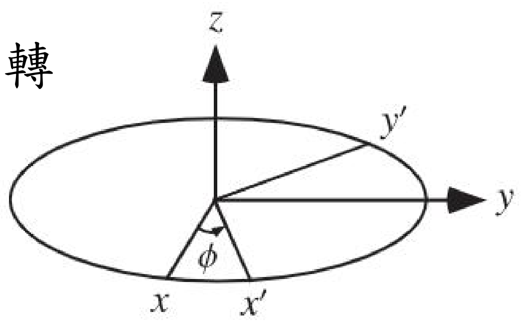
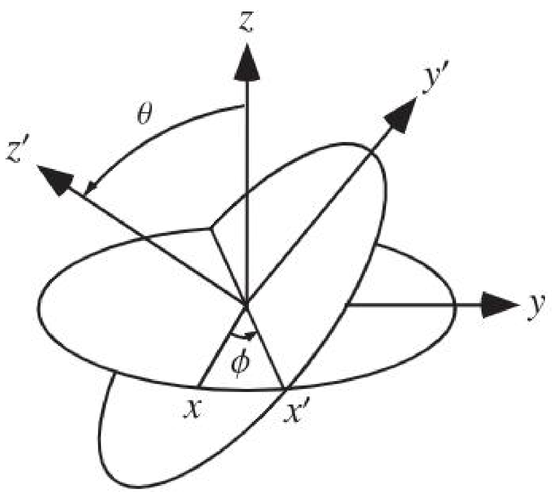


Fig. 9.29

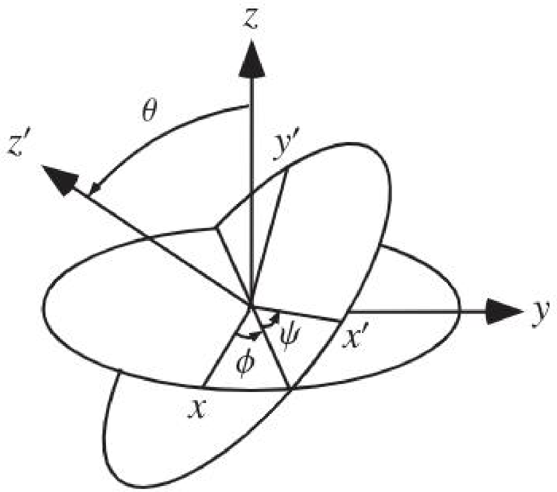
D



C



B



$$\mathbf{D} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[HW] 歐拉角旋轉矩陣：

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

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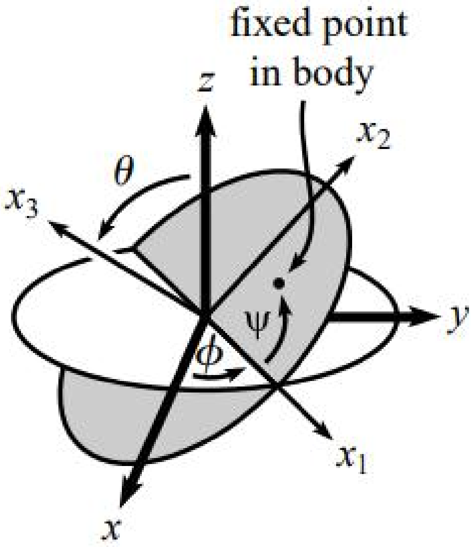
重對稱陀螺 Heavy Symmetric Top (1/8)

$$\boldsymbol{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$



有重力、$I_1 = I_2 \equiv I$

1. **力矩分析**：寫出角速度向量$\boldsymbol{\omega} = \dot{\psi}\hat{\mathbf{x}}_3 + (\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1)$.

$$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x}}_3 + \sin\theta\hat{\mathbf{x}}_2 \quad \Longrightarrow \quad \boldsymbol{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{x}}_3 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_2 + \dot{\theta}\hat{\mathbf{x}}_1$$

$\dot{\beta} = \dot{\psi} + \dot{\phi}\cos\theta$ 繞 3 方向轉

寫出角動量向量：$\mathbf{L} = I_3\dot{\beta}\hat{\mathbf{x}}_3 + I\dot{\phi}\sin\theta\hat{\mathbf{x}}_2 + I\dot{\theta}\hat{\mathbf{x}}_1$.

力矩公式：$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I_3\frac{d\dot{\beta}}{dt}\hat{\mathbf{x}}_3 + I\frac{d(\dot{\phi}\sin\theta)}{dt}\hat{\mathbf{x}}_2 + I\frac{d\dot{\theta}}{dt}\hat{\mathbf{x}}_1 + I_3\dot{\beta}\frac{d\hat{\mathbf{x}}_3}{dt} + I\dot{\phi}\sin\theta\frac{d\hat{\mathbf{x}}_2}{dt} + I\dot{\theta}\frac{d\hat{\mathbf{x}}_1}{dt}$.

$$\Longrightarrow \frac{d\mathbf{L}}{dt} = I_3\ddot{\beta}\hat{\mathbf{x}}_3 + \left(I\ddot{\phi}\sin\theta + 2I\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\beta}\dot{\theta}\right)\hat{\mathbf{x}}_2 + \left(I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\beta}\dot{\phi}\sin\theta\right)\hat{\mathbf{x}}_1.$$

[BONUS] 用幾何方式簡單說明此公式

$$\frac{d\hat{\mathbf{x}}_3}{dt} = -\dot{\theta}\hat{\mathbf{x}}_2 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_1,$$
$$\frac{d\hat{\mathbf{x}}_2}{dt} = \dot{\theta}\hat{\mathbf{x}}_3 - \dot{\phi}\cos\theta\hat{\mathbf{x}}_1,$$
$$\frac{d\hat{\mathbf{x}}_1}{dt} = -\dot{\phi}\sin\theta\hat{\mathbf{x}}_3 + \dot{\phi}\cos\theta\hat{\mathbf{x}}_2.$$

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重對稱陀螺 Heavy Symmetric Top (2/8)

[HW] 試算在 $\dot{\theta}=0$ 時的運動方程式長什麼樣子，還有此時陀螺的運動是什麼樣子？

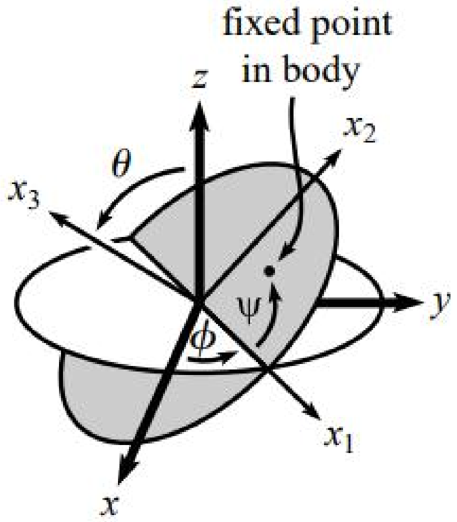
$$\ddot{\phi}=0 \Rightarrow \Omega \equiv \dot{\phi}=const.$$

陀螺的質心會繞 z 軸做水平的等速率圓周運動。

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mgl = 0$$

$$\Rightarrow \Omega_{\pm} = \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4IMgl\cos\theta}}{2I\cos\theta}$$

存在的條件：$\omega_3^2 \geq 4IMgl\cos\theta/I_3^2$，$\Omega_{\pm}$ 是陀螺的**進動角速率**。我們稱 Ω_+ 陀螺的「快進動角速度」，Ω_- 陀螺的「慢進動角速度」。



中央公園 Jan 3

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重對稱陀螺 Heavy Symmetric Top (3/8)

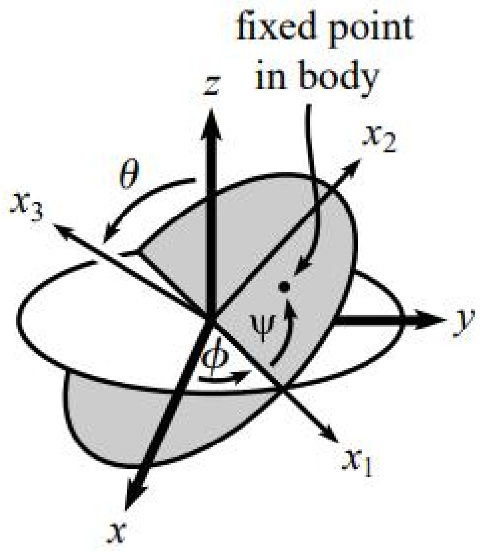
實際施加在陀螺上面的力矩指向 1 分量：

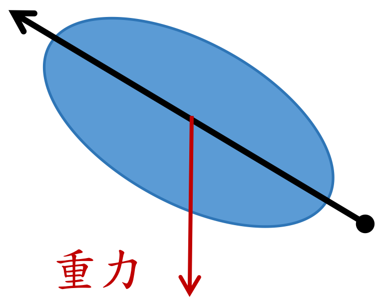
$$\frac{d\mathbf{L}}{dt} = I_3\ddot{\beta}\hat{\mathbf{x}}_3 + \left(I\ddot{\phi}\sin\theta + 2I\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\beta}\dot{\theta}\right)\hat{\mathbf{x}}_2 + \left(I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\beta}\dot{\phi}\sin\theta\right)\hat{\mathbf{x}}_1.$$

從 3 分量可以得到 $\ddot{\beta} = 0 \Longrightarrow \dot{\beta} = \text{const.} \equiv \omega_3$

$$I\ddot{\phi}\sin\theta + \dot{\theta}(2I\dot{\phi}\cos\theta - I_3\omega_3) = 0,$$

$$(Mg\ell + I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi})\sin\theta = I\ddot{\theta}.$$





其實力矩只在 x1 方向不為 0

物理意義：
(1) θ: 章動（nutation）
(2) ϕ: 質心（precession）進動
(3) ψ: 陀螺自轉

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重對稱陀螺 Heavy Symmetric Top (4/8)

Thm. 旋轉提供的動能為 $T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$

pf. 把物體切成很多份：$(dm)\, v^2/2 \;=\; dm\, |\boldsymbol{\omega} \times \mathbf{r}|^2/2.$

$$\boldsymbol{\omega} \times \mathbf{r} \;= (\omega_2 z - \omega_3 y)\hat{\mathbf{x}} + (\omega_3 x - \omega_1 z)\hat{\mathbf{y}} + (\omega_1 y - \omega_2 x)\hat{\mathbf{z}}.$$

$$|\boldsymbol{\omega} \times \mathbf{r}|^2 = (\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2$$

經過計算可以發現：

$$T = \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot \begin{pmatrix} \int (y^2 + z^2) & -\int xy & -\int zx \\ -\int xy & \int (z^2 + x^2) & -\int yz \\ -\int zx & -\int yz & \int (x^2 + y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \qquad (9.15)$$

重對稱陀螺 Heavy Symmetric Top (5/8)

2. **拉格朗日方法** $\mathcal{L}=T-V$： $V=Mg\ell\cos\theta,\quad T=\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}=\frac{1}{2}I_3(\dot\psi+\dot\phi\cos\theta)^2+\frac{1}{2}I(\dot\phi^2\sin^2\theta+\dot\theta^2).$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi}=\frac{\partial\mathcal{L}}{\partial\psi}\quad\Longrightarrow\quad\frac{d}{dt}(\dot\psi+\dot\phi\cos\theta)=0.$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi}=\frac{\partial\mathcal{L}}{\partial\phi}\quad\Longrightarrow\quad\frac{d}{dt}\left(I_3\omega_3\cos\theta+I\dot\phi\sin^2\theta\right)=0,$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta}=\frac{\partial\mathcal{L}}{\partial\theta}\quad\Longrightarrow\quad I\ddot\theta=(Mg\ell+I\dot\phi^2\cos\theta-I_3\omega_3\dot\phi)\sin\theta.$$

系統的守恆量：當拉格朗日量和一物理量 μ 無關時（即 $\partial\mathcal{L}/\partial\mu=0$），
此物理量 μ 所對應的「**共軛動量**」$\partial\mathcal{L}/\partial\dot\mu$ 守恆。

這裡守恆的共軛動量是 $\mathbf{x}_3$、z 方向的角動量：

$$L_z=L_3\cos\theta+L_2\sin\theta=(I_3\omega_3)\cos\theta+(I\dot\phi\sin\theta)\sin\theta$$

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重對稱陀螺 Heavy Symmetric Top (6/8)

[BONUS]
(I) 快 / 慢進動角速度：
一個質量為 M 的對稱陀螺，其質心與支點相距 ℓ 。相對於支點的轉動慣量為 $I = I_1 = I_2$，以及 I_3 。此陀螺繞其對稱軸以角頻率 ω_3 旋轉，且假設
(1) 質心繞鉛直軸作圓周運動。
(2) 對稱軸與鉛直方向之間夾一個固定角度 θ（見圖 9.31）。

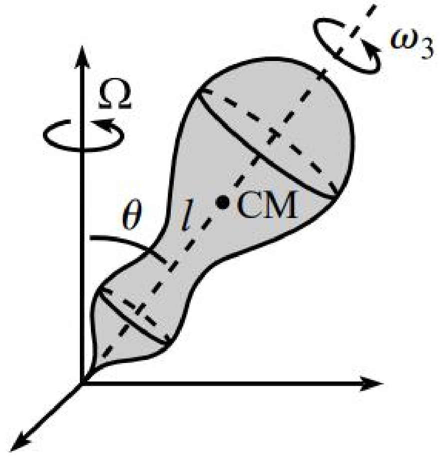

Fig. 9.31

(a) 假設由 ω_3 所造成的角動量遠大於此問題中任何其他角動量，求進動頻率 Ω 的近似表達式。
(b) 現在精確地解這個問題。也就是說，考慮所有角動量後，求出 Ω。

Sol. (參考答案) **[Check]** 慢進動角速度 = (a) 的近似解

(a) $\Omega = \dfrac{Mg\ell}{I_3\omega_3}$. (b) $\Omega_\pm = \dfrac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \dfrac{4MIg\ell\cos\theta}{I_3^2\omega_3^2}}\right)$.

重對稱陀螺 Heavy Symmetric Top (7/8)

(II) 陀螺的章動（nutation）：
(1) θ 可以變化但變化量不會太大
(2) 陀螺轉速很快：$\omega_3 \gg \dot{\phi},\ \dot{\theta}$

$$I\ddot{\phi}\sin\theta + \dot{\theta}(2I\dot{\phi}\cos\theta - I_3\omega_3) = 0,$$
$$(Mg\ell + I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi})\sin\theta = I\ddot{\theta}.$$

用 (2)：

$$I\ddot{\phi}\sin\theta - \dot{\theta}I_3\omega_3 = 0,$$

[HW] 驗算

$$(Mg\ell - I_3\omega_3\dot{\phi})\sin\theta = I\ddot{\theta}.$$

$$\Longrightarrow \quad \frac{d^2\dot{\phi}}{dt^2} + \omega_n^2(\dot{\phi} - \Omega_s) = 0,$$

$$I\dddot{\phi}\sin\theta + I\ddot{\phi}\dot{\theta}\cos\theta - \dot{\theta}\ddot{\theta}I_3\omega_3 = 0 \qquad \omega_n \equiv \frac{I_3\omega_3}{I} \quad \text{and} \quad \Omega_s = \frac{Mg\ell}{I_3\omega_3}$$

解出：$\dot{\phi}(t) = \Omega_s + A\cos(\omega_n t + \gamma), \quad \phi(t) = \Omega_s t + \left(\frac{A}{\omega_n}\right)\sin(\omega_n t + \gamma),$

[HW]：$\dot{\theta}(t) = -\left(\frac{I\sin\theta}{I_3\omega_3}\right)A\omega_n\sin(\omega_n t + \gamma) \Longrightarrow \theta(t) = B + \left(\frac{A}{\omega_n}\sin\theta_0\right)\cos(\omega_n t + \gamma),$

轉得越快，章動越小

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重對稱陀螺 Heavy Symmetric Top (8/8)

(III) 陀螺受到干擾（sideway kick）

假設：

(1) 陀螺的質心做水平等速率圓周運動：$\theta = \theta_0$、$\dot{\phi} = \Omega_s$（沒有初始章動）

(2) 輕輕踢陀螺，使得 Ω_s 變成 $\Omega_s + \Delta\Omega$

(1’) θ 可以變化但變化量不會太大

(2’) 陀螺轉速很快：$\omega_3 \gg \dot{\phi},\ \dot{\theta}$

接下來 $\theta(t)$ 、$\phi(t)$ 會怎麼改變？

Sol. 章動分析： $\phi(t) = \Omega_s t + \left(\frac{A}{\omega_n}\right)\sin(\omega_n t + \gamma), \quad \theta(t) = B + \left(\frac{A}{\omega_n}\sin\theta_0\right)\cos(\omega_n t + \gamma),$

$\dot{\theta}(0) = 0$：$\gamma = 0$ or π

$\dot{\phi}(0) = \Omega_s + \Delta\Omega$：$A = \Delta\Omega$

$\theta(0) = \theta_0$： $B = \theta_0 - (\Delta\Omega/\omega_n)\sin\theta_0$

$$\phi(t) = \Omega_s t + \left(\frac{\Delta\Omega}{\omega_n}\right)\sin\omega_n t,$$

$$\theta(t) = \theta_0 - \left(\frac{\Delta\Omega}{\omega_n}\sin\theta_0\right)(1 - \cos\omega_n t).$$